Velocity Dispersions; Boltzmann and Jeans equations; and applications

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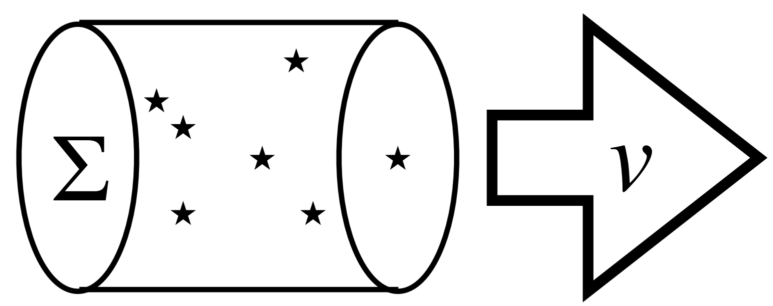
Stellar populations as a collisionless fluid

- Today we shall develop a mathematical framework for treating the motions of stars *statistically*
- The Galaxy is too complex to model on the level of individual stars ($\sim 10^{11}$ bodies)
- Instead, we take a statistical approach, working with the average motions
- An important simplification is that we assume a population of stars is *collisionless*
  - *"collision"* here does not imply physical touching
  - rather, it refers to gravitational action at a distance (think of "collisions" between charged particles)
  - the important thing is that stars' velocities are not impulsively changed as a result of interactions with each other

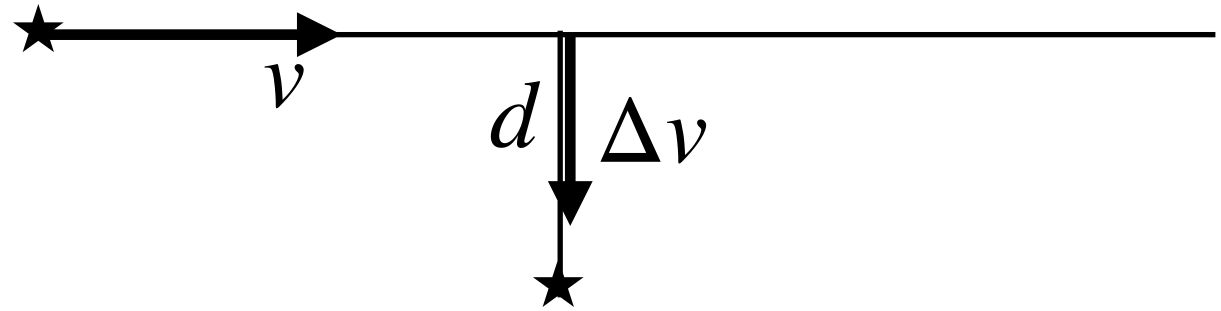
Justifying the collisionless assumption

- Consider a star moving at velocity v through a background population of other stars, with number density n stars per pc^3
- The rate at which it encounters stars within a distance d is given by $n\Sigma v$, where $\Sigma = \pi d^2$



- The velocity will be similar to the velocity dispersion of stars in the Solar neighbourhood, $\sim 30\,\mathrm{km\,s^{-1}}$
- How close must two stars be to significantly affect each others' orbits in the Galaxy?
- How often does this occur?

Justifying the collisionless assumption



- Consider one star flying past another of mass M at a distance d. How big a kick Δv do they give each other?
- Working in an impulse approximation (assume velocity changes instantaneously at the point of closest approach):
  - acceleration is $F = \dfrac{GM}{d^2}$
  - timescale is $\Delta t = d/v$
  - then $\Delta v = F\Delta t = \dfrac{GM}{dv}$
- If we want $\Delta v \sim v$ that means $(\Delta v)^2 \sim \dfrac{GM}{d}$
- Now if $v \sim 30\,\mathrm{km\,s^{-1}}$, we must have $d = 1\,\mathrm{au} = 5 \times 10^{-6}\,\mathrm{pc}$
- This is tiny compared to the mean distance between stars of ~ 1 pc

Justifying the collisionless assumption

- Now we have $v \sim 30\,\mathrm{km\,s^{-1}} \approx 30\,\mathrm{pc\,Myr^{-1}}$, $d = 5 \times 10^{-6}\,\mathrm{pc}$, and $n \sim 0.1\,\mathrm{pc^{-3}}$
- Thus the encounter rate is $n\Sigma v = 0.1 \times 3 \times (5 \times 10^{-6})^2 \times 30\,\mathrm{Myr^{-1}} \sim 2 \times 10^{-10}\,\mathrm{Myr^{-1}}$
- Thus star–star encounters are negligible
- On the other hand, if we interact with a large Giant Molecular Cloud of mass $\sim 10^6\,\mathrm{M_\odot}$, we would only have to pass within 5 pc to get an equivalent Δv
- Thus, occasional encounters with GMCs can change stars' velocities
- Q: what about in a cluster, $\sigma \sim 1\,\mathrm{km\,s^{-1}}$, $n \sim 1000\,\mathrm{pc^{-3}}$?

We now treat stars as collisionless

- We assume that stars are accelerating under the (smooth) Galactic potential $\psi(t, x, y, z)$
- Star–star encounters are neglected
- We also will neglect the GMCs, and complications like spiral arms
- Later we will assume certain symmetries such as time-invariance and axisymmetry

The phase space density

- We assume that stars are accelerating under the (smooth) Galactic potential $\psi(t, x, y, z)$
- Rather than solving the equations of motion for each star (*cf*. last lecture), we work with the density distribution of stars
- We have, at each point in the Galaxy, a number density $n(t, x, y, z)$ measured in stars per pc^{-3}
- This refers to a tracer star population. n can be *vastly different* for different types of stars (young OB stars, M dwarfs, red giants, white dwarfs, black holes,…)
- It loosely relates to the matter density $\rho(t, x, y, z)$ which gives rise to the potential $\psi(t, x, y, z)$
- $\rho = \sum_{\text{all stellar pops}} M_\star n_\star$
- Here, we will treat n and the stellar populations as *passive:* they do not themselves affect ρ

The phase space density

- The number density $n(t, x, y, z)$ is not sufficient to describe stellar kinematics: we need a description of velocities
- Because the stars are collisionless, the velocity distributions can be quite complex. We define a velocity distribution $f_v(u, v, w)$ where $u = \dot{x}$, $v = \dot{y}$ and $w = \dot{z}$
- This defines a density in velocity space
- However, the velocity distribution also depends on the location in physical (x, y, z) space, as well as on time
- Hence, the full description of the stellar population is given by the *phase space density* $f(t, x, y, z, u, v, w)$ which has units particles per cubic parsec per cubic km s^{-1}

Motion in phase space and integrals of motion

- An individual body's motion is totally determined by the position in phase space (x, y, z, u, v, w) and the potential $\psi(t, x, y, z)$:
- $\frac{\mathrm{d}}{\mathrm{d}t}(x, y, z, u, v, w) = \left(u, v, w, -\frac{\partial\psi}{\partial x}, -\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial z}\right)$
- This defines a trajectory that will eventually visit all points in a subset of phase space. Some phase space may be excluded
- Some restrictions arise from integrals of motion: conserved quantities
- Some of these arise from physical symmetries (Noether's theorem)
  - Time invariance gives conservation of energy
    $E = \psi(x, y, z) + \frac{1}{2}\left(u^2 + v^2 + w^2\right)$
  - Axisymmetry gives conservation of the z-component of angular momentum
    $L_z = xv - yu$
  - Spherical symmetry gives conservation of the full angular momentum vector $\mathbf{L}$
  - Translational symmetry gives conservation of linear momentum

Moments of the phase space density

- The phase space density itself is too complex a function to deal with directly
- If we neglect time, there are six dimensions. To get statistics on $f(x, y, z, u, v, w)$, we could imagine dividing up phase space into cubic bins $\Delta x \Delta y \Delta z \Delta u \Delta v \Delta w$
- To get an average of only one star per bin, we would need 10^6 stars. To get good stats ($N = 100$ gives a 10% relative error), we would need 10^8 stars
- Therefore, we restrict attention to the Solar neighbourhood (one big “bin” in (x, y, z)) and take moments over the velocity dimensions
- “moments” are means and dispersions

Moments of the phase space density

- In general, the moments are
- $$\iiint_{-\infty}^{\infty} u^i v^j w^k f(t, x, y, z, u, v, w) \mathrm{d}u \mathrm{d}v \mathrm{d}w$$
- i, j, k are integers ≥ 0
- The *order* of the moment is $i + j + k \geq 0$
- We will use moments of order 0, 1 and 2
- We will also use *central moments* of order 2

The zeroth-order moment

- $\iiint_{-\infty}^{\infty} u^i v^j w^k f(t, x, y, z, u, v, w) \mathrm{d}u \mathrm{d}v \mathrm{d}w$
- The zeroth-order moment has $i = j = k = 0$
- Thus, it is $\iiint f(t, x, y, z, u, v, w) \mathrm{d}u \mathrm{d}v \mathrm{d}w$
- *I.e.*, pick a point in space and add up all stars of all velocities: this is just the number density $n(t, x, y, z)$

The first-order moments

- $$\iiint_{-\infty}^{\infty} u^i v^j w^k f(t, x, y, z, u, v, w) \mathrm{d}u \mathrm{d}v \mathrm{d}w$$
- The three first-order moments each have one of i, j or k equal to unity
- They relate to the mean velocities, *e.g.*,
- $$n \langle u \rangle = \iiint u f \mathrm{d}u \mathrm{d}v \mathrm{d}w$$
- *Cf*. the mean of a discrete distribution:
- $$\langle x \rangle = \frac{\sum_i x_i n_i}{\sum_i n_i}$$

The second-order moments

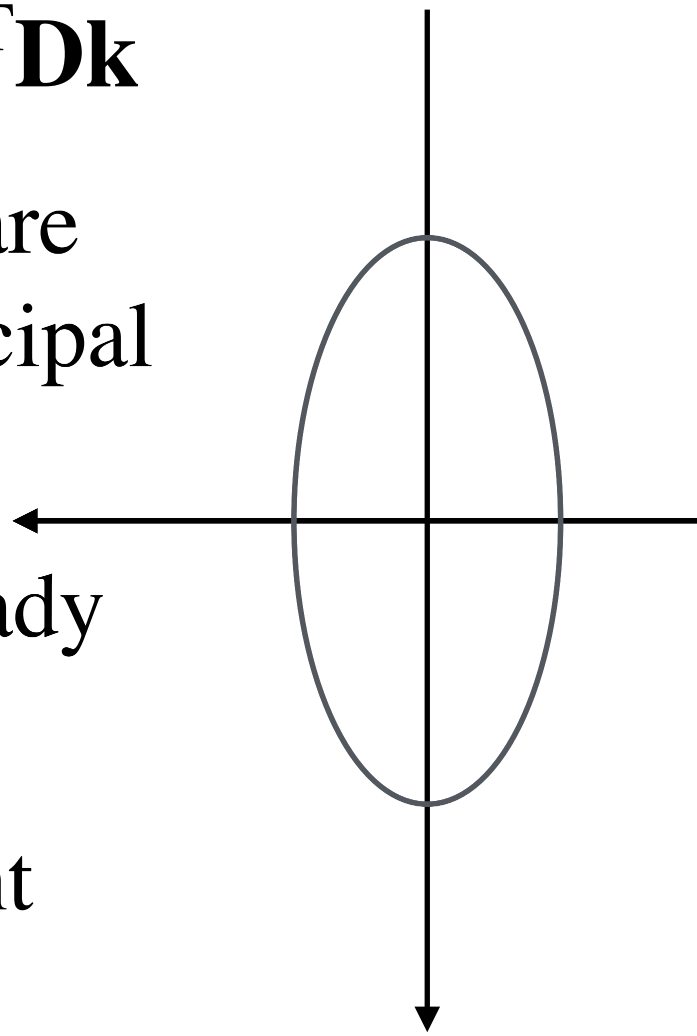
- $\iiint_{-\infty}^{\infty} u^i v^j w^k f(t, x, y, z, u, v, w) \mathrm{d}u \mathrm{d}v \mathrm{d}w$
- There are six second-order moments, with $i + j + k = 2$
- They take the form
- $n \left\langle u^2 \right\rangle = \iiint u^2 f \mathrm{d}u \mathrm{d}v \mathrm{d}w$
- or
- $n \left\langle uv \right\rangle = \iiint uv f \mathrm{d}u \mathrm{d}v \mathrm{d}w$
- These can be measured observationally, but are not so physically useful: the second-order moments should be telling us about the spread of a distribution, insensitive to its peak

The second-order central moments

- Therefore, we work instead with the second-order *central moments*, taken relative to the mean:
- $D_{uu} = \left\langle (u - \langle u \rangle)^2 \right\rangle = \left\langle u^2 \right\rangle - \langle u \rangle^2$
- $D_{uv} = \left\langle (u - \langle u \rangle)(v - \langle v \rangle) \right\rangle = \langle uv \rangle - \langle u \rangle \langle v \rangle \; etc.$
- These central moments measure the shape of a distribution, independent of its position
- The above moments are two of the six independent elements of the symmetric dispersion matrix **D**
- $$\mathbf{D} = \begin{pmatrix} D_{uu} & D_{uv} & D_{uw} \\ D_{vu} & D_{vv} & D_{vw} \\ D_{wu} & D_{wv} & D_{ww} \end{pmatrix} = \frac{1}{n} \iiint \left(\mathbf{v} - \langle \mathbf{v} \rangle\right) \left(\mathbf{v} - \langle \mathbf{v} \rangle\right)^{\mathrm{T}} f \mathrm{d}^3\mathbf{v}$$

Velocity dispersions

- The diagonal elements of **D** are the velocity dispersions in the *x*, *y* and *z* directions
- $E.g., D_{uu} = \sigma_u^2$
- If we want the dispersion along an arbitrary direction **k**, we use the *quadratic form* $\sigma_k^2 = \mathbf{k}^{\mathrm{T}}\mathbf{D}\mathbf{k}$
- **D** is symmetric and therefore its eigenvectors are orthogonal. These eigenvectors define the principal axes of a 3D gaussian velocity distribution
- If **D** is already diagonal, its principal axes already align with the (x, y, z) axes
- If not, the velocity ellipsoid points in a different direction (the *vertex deviation*)



Measuring the velocity dispersion

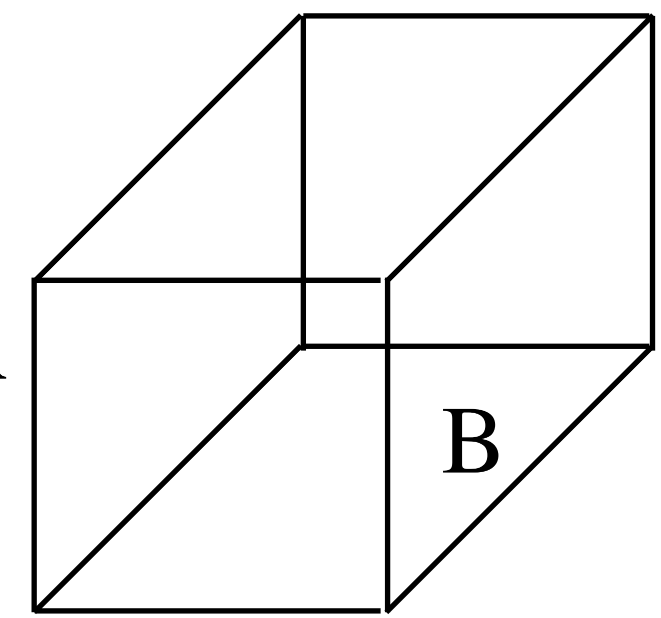
- Recall that we have defined these moments at a fixed point (x, y, z)
- In practice, we need to measure stellar kinematics in a finite volume $\mathscr{V}$. This should be small enough that the stellar velocities are statistically the same over the whole volume, *i.e.*, smaller than the large-scale velocity structures…
- …but big enough to give a decent-sized sample of stars $N \gtrsim 100$; recall $N = 100$ gives a 10% error on the mean
- Then we simply have *e.g.*, the number density $n \approx N/\mathscr{V}$
- $\langle u \rangle \approx \frac{1}{N} \sum_{i=1}^{N} u_i$
- $D_{uu} \approx \frac{1}{N} \sum_{i=1}^{N} \left(u - \langle u \rangle\right)^2$

The Boltzmann and Jeans equations

- We shall now derive the *collisionless Boltzmann equation*, governing the evolution of the phase space density
- We shall then take moments of this equation, deriving the *Jeans equations* which relate components of the means and the velocity dispersion to each other, and to the potential
- This will allow us to learn about the properties of the Galactic potential, without actually measuring the direct accelerations of stars. However, we will have to make some simplifying assumptions
- These equations have analogies with the equations of fluid dynamics, for example the continuity equation and the Euler equation

The continuity equation (in 6D phase space)

- We recall that our "fluid" of stars is collisionless, so stars only accelerate under the smooth Galactic potential $\psi(t, x, y, z)$



- We imagine a small 6D hypercube
- The flow into this volume in the x-direction is $\dot{f}_A = f_A \dot{x}_A \mathrm{d}y\mathrm{d}z\mathrm{d}u\mathrm{d}v\mathrm{d}w$
- The flow out is $\dot{f}_B = f_B \dot{x}_B \mathrm{d}y\mathrm{d}z\mathrm{d}u\mathrm{d}v\mathrm{d}w$

- Thus, the net flow in the x-direction is
- $$\left(\frac{\partial f}{\partial t}\right)_{x \text{ contribution}} = (f_A \dot{x}_A - f_B \dot{x}_B)\mathrm{d}y\mathrm{d}z\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{\partial (f\dot{x})}{\partial x}\mathrm{d}y\mathrm{d}z\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

The continuity equation (in 6D phase space)

- We have similar contributions in the other five dimensions, so the total rate of change of *f* is, in the absence of collisions,
- $$\frac{\partial f}{\partial t} = -\frac{\partial(f\dot{x})}{\partial x} - \frac{\partial(f\dot{y})}{\partial y} - \frac{\partial(f\dot{z})}{\partial z} - \frac{\partial(f\dot{u})}{\partial u} - \frac{\partial(f\dot{v})}{\partial v} - \frac{\partial(f\dot{w})}{\partial w}$$
- Now, $u = \dot{x}$ by definition; $\frac{\partial \dot{x}}{\partial x} = \frac{\partial u}{\partial x} = 0$ as these are independent dimensions; $\frac{\partial \dot{u}}{\partial u} = 0$ as the acceleration depends only on position; so we have
- $$\frac{\partial f}{\partial t} + +u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + \dot{u}\frac{\partial f}{\partial u} + \dot{v}\frac{\partial f}{\partial v} + \dot{w}\frac{\partial f}{\partial w} = 0$$
- This we can write $\nabla_{6D} \cdot (f\mathbf{p}) + \dot{f} = 0$ where $\mathbf{p} = (\mathbf{x}, \mathbf{v})$ and ∇_{6D} is a six-dimensional divergence operator
- *Cf*. the regular 3D *continuity equation* of hydrodynamics: $\nabla \cdot (\rho\mathbf{u}) + \dot{\rho} = 0$

The Collisionless Boltzmann Equation (CBE)

- Recall:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + \dot{u}\frac{\partial f}{\partial u} + \dot{v}\frac{\partial f}{\partial v} + \dot{w}\frac{\partial f}{\partial w} = 0$$

- For the systems we are interested in, recall that the accelerations are only produced by the large-scale potential $\psi(t, x, y, z)$:

$$(u, v, w) = \left(-\frac{\partial \psi}{\partial x}, -\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial z}\right)$$

- Substituting, we get the *Collisionless Boltzmann equation (CBE):*

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\frac{\partial f}{\partial u} - \frac{\partial \psi}{\partial y}\frac{\partial f}{\partial v} - \frac{\partial \psi}{\partial z}\frac{\partial f}{\partial w} = 0$$

- This tells us (implicitly) how the phase space density evolves under the influence of the galactic potential

The Collisionless Boltzmann Equation (CBE)

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\frac{\partial f}{\partial u} - \frac{\partial \psi}{\partial y}\frac{\partial f}{\partial v} - \frac{\partial \psi}{\partial z}\frac{\partial f}{\partial w} = 0$$

- The form above holds for a Cartesian coordinate system
- In galactic dynamics, we often find it more useful to work in cylindrical polar coordinates to exploit axisymmetry
- The CBE then takes the form

$$\frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + v_z\frac{\partial f}{\partial z} +$$

Coriolis

$$\left(\frac{v_\phi^2}{R} - \frac{\partial \psi}{\partial R}\right)\frac{\partial f}{\partial v_R} - \left(\frac{v_R v_\phi}{R} + \frac{1}{R}\frac{\partial \psi}{\partial \phi}\right)\frac{\partial f}{\partial v_\phi} - \frac{\partial \psi}{\partial z}\frac{\partial f}{\partial v_z} = 0$$

Centrifugal

- where $\left(v_R, v_\phi, v_z\right)$ are the radial, azimuthal and vertical velocities

The Jeans Equations

- Recall that the phase space density itself is too cumbersome to work with
- The CBE itself is not therefore always very useful
- However, if we take moments of the CBE, we get some very useful equations: The *Jeans equations*
- Recall that the n^{th}-order moments of f are obtained by multiplying by $u^i v^j w^k$ and integrating over velocity space, where $i + j + k = n \geq 0$
- The (single) 0^{th}-order moment is the number density
- The (three) 1^{st}-order moments are the average velocities
- The (six) 2^{nd}-order moments relate to the dispersion matrix

The First Jeans Equation

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\frac{\partial f}{\partial u} - \frac{\partial \psi}{\partial y}\frac{\partial f}{\partial v} - \frac{\partial \psi}{\partial z}\frac{\partial f}{\partial w} = 0$$

- We take the 0$^{\mathrm{th}}$-order moment of the CBE: integrate dudvdw

$$\iiint \frac{\partial f}{\partial t}\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{\partial}{\partial t}\iiint f\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{\partial n}{\partial t}$$

$$\iiint u\frac{\partial f}{\partial x}\mathrm{d}u\mathrm{d}v\mathrm{d}w = \iiint \frac{\partial (uf)}{\partial x}\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{\partial}{\partial x}\iiint uf\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{\partial n\left\langle u\right\rangle}{\partial x}$$

$$\iiint \frac{\partial \psi}{\partial x}\frac{\partial f}{\partial u}\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{\partial \psi}{\partial x}\iiint \frac{\partial f}{\partial u}\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{\partial \psi}{\partial x}\iint \left[f\right]_{-\infty}^{\infty}\mathrm{d}u\mathrm{d}v\mathrm{d}w = 0$$

- Thus:

$$\frac{\partial n}{\partial t} + \frac{\partial\left(n\left\langle u\right\rangle\right)}{\partial x} + \frac{\partial\left(n\left\langle v\right\rangle\right)}{\partial y} + \frac{\partial\left(n\left\langle w\right\rangle\right)}{\partial z} = 0$$

The First Jeans Equation

$$\frac{\partial n}{\partial t} + \frac{\partial\left(n\langle u\rangle\right)}{\partial x} + \frac{\partial\left(n\langle v\rangle\right)}{\partial y} + \frac{\partial\left(n\langle w\rangle\right)}{\partial z} = 0$$

- This may be written vectorially:

$$\dot{n} + \nabla\cdot\left(n\langle\mathbf{v}\rangle\right) = 0$$

- in which form it is analogous to the 3D continuity equation of hydrodynamics:

$$\dot{\rho} + \nabla\cdot\left(\rho\mathbf{v}\right) = 0$$

- This means that the number density changes only because of the flow of particles into or out of a point
- This brings up an implicit assumption we have made: stars are neither created nor destroyed

The Second Jeans Equations

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\frac{\partial f}{\partial u} - \frac{\partial \psi}{\partial y}\frac{\partial f}{\partial v} - \frac{\partial \psi}{\partial z}\frac{\partial f}{\partial w} = 0$$

- There are three of these: one for each of the 1st-order moments: multiply by *u*, *v* or *w* and integrate

$$\frac{\partial\left(n\langle u\rangle\right)}{\partial t} + \frac{\partial\left(n\left\langle u^2\right\rangle\right)}{\partial x} + \frac{\partial\left(n\langle uv\rangle\right)}{\partial y} + \frac{\partial\left(n\langle uw\rangle\right)}{\partial z} + n\frac{\partial \psi}{\partial x} = 0$$

and similarly for the *v* and *w* moments

The Third Jeans Equations

$$\frac{\partial\left(n\langle u\rangle\right)}{\partial t}+\frac{\partial\left(n\left\langle u^2\right\rangle\right)}{\partial x}+\frac{\partial\left(n\langle uv\rangle\right)}{\partial y}+\frac{\partial\left(n\langle uw\rangle\right)}{\partial z}+n\frac{\partial\psi}{\partial x}=0$$

- There are three of these, obtained by re-writing the second Jeans equations in terms of the dispersion matrix:

$$\frac{\partial\left(n\langle u\rangle\right)}{\partial t}=n\frac{\partial\langle u\rangle}{\partial t}+\langle u\rangle\frac{\partial n}{\partial t}$$

$$\frac{\partial\left(n\left\langle u^2\right\rangle\right)}{\partial x}=\frac{\partial\left(nD_{uu}+n\langle u\rangle^2\right)}{\partial x}=\frac{\partial\left(nD_{uu}\right)}{\partial x}+n\langle u\rangle\frac{\partial\langle u\rangle}{\partial x}+\langle u\rangle\frac{\partial\left(n\langle u\rangle\right)}{\partial x}$$

- From these we obtain

$$\frac{\partial\langle u\rangle}{\partial t}+\langle u\rangle\frac{\partial\langle u\rangle}{\partial x}+\langle v\rangle\frac{\partial\langle u\rangle}{\partial y}+\langle w\rangle\frac{\partial\langle u\rangle}{\partial z}+$$

$$\frac{1}{n}\left[\frac{\partial\left(nD_{uu}\right)}{\partial x}+\frac{\partial\left(nD_{uv}\right)}{\partial y}+\frac{\partial\left(nD_{uw}\right)}{\partial z}\right]=-\frac{\partial\psi}{\partial x}$$

The Third Jeans Equations

$$\frac{\partial \langle u \rangle}{\partial t} + \langle u \rangle \frac{\partial \langle u \rangle}{\partial x} + \langle v \rangle \frac{\partial \langle u \rangle}{\partial y} + \langle w \rangle \frac{\partial \langle u \rangle}{\partial z} +$$

$$\frac{1}{n}\left[\frac{\partial \left(nD_{uu}\right)}{\partial x} + \frac{\partial \left(nD_{uv}\right)}{\partial y} + \frac{\partial \left(nD_{uw}\right)}{\partial z}\right] = -\frac{\partial \psi}{\partial x}$$

- Everything here is observable, except for the potential gradient and the time derivative!
- As the first Jeans equation, this may be written vectorially:

$$\langle \dot{\mathbf{v}} \rangle + \langle \mathbf{v} \rangle \cdot \nabla \langle \mathbf{v} \rangle + \frac{1}{n}\nabla \cdot (n\mathbf{D}) = -\mathbf{F_g}$$

- This is analogous to the Euler equation of hydrodynamics for a frictionless fluid:

$$\dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\rho}\nabla p = -\mathbf{F_g}$$

- If we recall the ideal gas law $p = \frac{k_B}{\mu m}\rho T$ we see that the velocity dispersion is playing the role of a "temperature"

The Jeans Equations in a cylindrical geometry

We will soon (for asymmetric drift) make use of the second Jeans equation in cylindrical polar coordinates:

$$\frac{\partial\left(n\left\langle v_R\right\rangle\right)}{\partial t}+\frac{\partial\left(n\left\langle v_R^2\right\rangle\right)}{\partial R}+\frac{1}{R}\frac{\partial\left(n\left\langle v_R v_\phi\right\rangle\right)}{\partial \phi}+$$

$$\frac{\partial\left(n\left\langle v_R v_z\right\rangle\right)}{\partial z}+\frac{n}{R}\left(\left\langle v_R^2\right\rangle-\left\langle v_\phi^2\right\rangle\right)+n\frac{\partial\psi}{\partial R}=0$$

Applications of the Jeans Equations

- We now have a set of equations relating velocity, velocity dispersion, number density, and the potential, together with their derivatives
- All of these things are observable, except for the potential, and any time derivatives
- We now proceed to use these equations to understand two phenomena:
  - the vertical structure of the Galactic disc
  - the asymmetric drift
- Each requires some simplifications to make the Jeans equations tractable
- Most importantly, we assume *time-invariance:* $\frac{\partial}{\partial t} = 0$

The vertical motions of stars

- We adopt the *plane-parallel approximation*, where we assume that the statistical properties of stellar motions only depend on the z-coordinate, *i.e.*, on the distance out of the Galactic midplane
- This is justified because the disc scale height ~ 300pc whereas the disc scale length is ~3kpc
- We also assume axisymmetry
- Thus we have:
  - $\frac{\partial}{\partial t} = 0$
  - $\frac{\partial}{\partial x} = 0$
  - $\frac{\partial}{\partial y} = 0$
- And we use the third Jeans equation, where only the z-component is non-zero

The vertical motions of stars

$$\frac{\partial \langle w \rangle}{\partial t} + \langle u \rangle \frac{\partial \langle w \rangle}{\partial x} + \langle v \rangle \frac{\partial \langle w \rangle}{\partial y} + \langle w \rangle \frac{\partial \langle w \rangle}{\partial z} +$$

$$\frac{1}{n}\left[\frac{\partial \left(nD_{wu}\right)}{\partial x} + \frac{\partial \left(nD_{wv}\right)}{\partial y} + \frac{\partial \left(nD_{ww}\right)}{\partial z}\right] = -\frac{\partial \psi}{\partial z}$$

- Since *x*, *y* and *t* derivatives are zero, this reduces to

$$\langle w \rangle \frac{\partial \langle w \rangle}{\partial z} + \frac{1}{n}\frac{\partial \left(nD_{ww}\right)}{\partial z} = -\frac{\partial \psi}{\partial z}$$

- We also assume $\langle w \rangle = 0$ [why?]:

$$\frac{1}{n}\frac{\partial \left(nD_{ww}\right)}{\partial z} = -\frac{\partial \psi}{\partial z}$$

- This is analogous to the equation for hydrostatic equilibrium in a plane-parallel atmosphere:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

The vertical motions of stars

$$\frac{1}{n}\frac{\partial\left(nD_{ww}\right)}{\partial z} = -\frac{\partial\psi}{\partial z}$$

- We next assume that $D_{ww} = \sigma_w^2$ is independent of z:

$$\frac{\partial\psi}{\partial z} = -\sigma_w^2\frac{\partial \ln n}{\partial z}$$

- Recall that all of these equations are true for any population of tracer stars. Different populations of stars have different velocity dispersions and densities, but they all feel the same potential

- Differentiating by z:

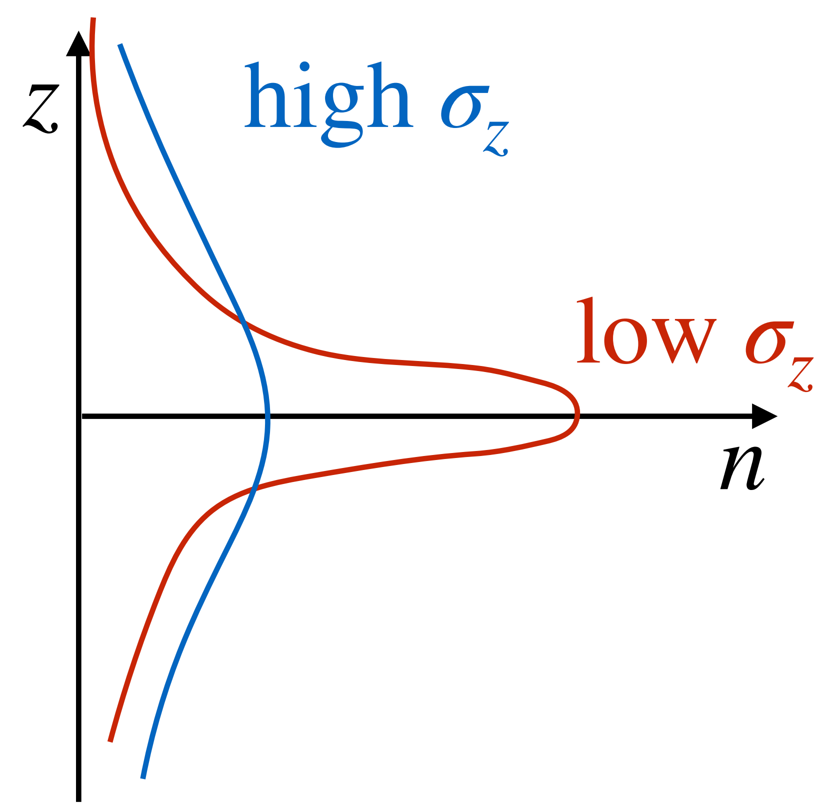
$$\frac{\partial^2\psi}{\partial z^2} = -\sigma_w^2\frac{\partial^2 \ln n}{\partial z^2}$$

- Recalling Poisson's Equation $\nabla^2\psi = \frac{\partial^2\psi}{\partial z^2} = 4\pi G\rho(z)$:

$$4\pi G\rho(z) = -\sigma_w^2\frac{\partial^2 \ln n}{\partial z^2}$$

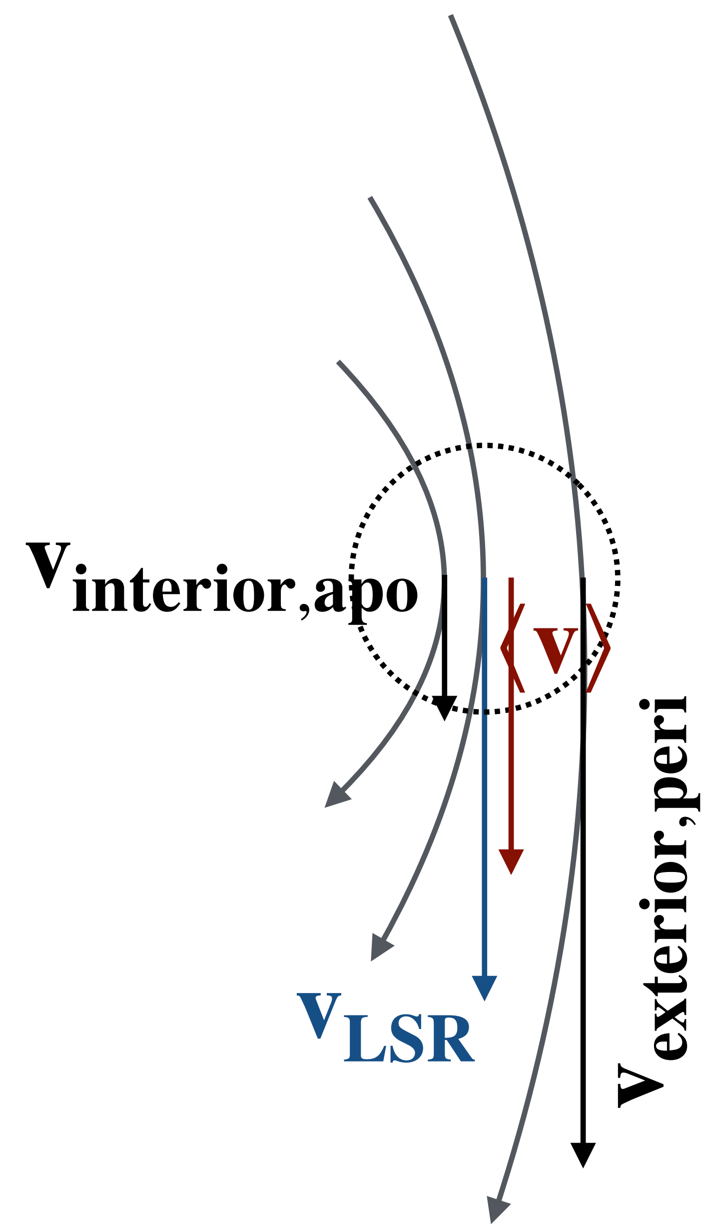
- And in the midplane, $z = 0$:

$$\rho(0) = -\frac{\sigma_w^2}{4\pi G}\left[\frac{\partial^2 \ln n}{\partial z^2}\right]_{z=0}$$



Asymmetric drift

- The mean orbital velocity is, in general, not the same as the velocity of a circular orbit (the LSR velocity)!
- The mean velocity is usually slower than the LSR velocity: this is called *asymmetric drift*
- Why is this?
    - there are more stars closer to the Galactic centre
    - stars move more slowly at orbital apocentre
    - so there are more stars at apocentre on interior orbits, than at pericentre on exterior orbits, and they spend more time there
    - Thus, mean velocity lags the circular velocity



Asymmetric Drift

- Here we use the second Jeans equation in cylindrical polar coordinates:

$$\frac{\partial\left(n\left\langle v_R\right\rangle\right)}{\partial t}+\frac{\partial\left(n\left\langle v_R^2\right\rangle\right)}{\partial R}+\frac{1}{R}\frac{\partial\left(n\left\langle v_R v_\phi\right\rangle\right)}{\partial\phi}+$$

$$\frac{\partial\left(n\left\langle v_R v_z\right\rangle\right)}{\partial z}+\frac{n}{R}\left(\left\langle v_R^2\right\rangle-\left\langle v_\phi^2\right\rangle\right)+n\frac{\partial\psi}{\partial R}=0$$

- We make the following assumptions:
  - $\frac{\partial}{\partial t}=0$
  - $\frac{\partial}{\partial\phi}=0$
  - No net flow of stars in R, meaning that $\left\langle v_R\right\rangle=0$
- The Jeans equation then becomes

$$\frac{\partial\left(n\left\langle v_R^2\right\rangle\right)}{\partial R}+\frac{\partial\left(n\left\langle v_R v_z\right\rangle\right)}{\partial z}+\frac{n}{R}\left(\left\langle v_R^2\right\rangle-\left\langle v_\phi^2\right\rangle\right)+n\frac{\partial\psi}{\partial R}=0$$

Asymmetric Drift

$$\frac{\partial\left(n\left\langle v_R^2\right\rangle\right)}{\partial R}+\frac{\partial\left(n\left\langle v_R v_z\right\rangle\right)}{\partial z}+\frac{n}{R}\left(\left\langle v_R^2\right\rangle-\left\langle v_\phi^2\right\rangle\right)+n\frac{\partial\psi}{\partial R}=0$$

- Now, stellar kinematics are only weakly dependent on R, but the number density has an e-folding length scale of ~3kpc. Hence we ignore the derivative of $\left\langle v_R^2\right\rangle$. We also ignore $\left\langle v_R v_z\right\rangle$:

$$\left\langle v_R^2\right\rangle\frac{\partial n}{\partial R}+\frac{n}{R}\left(\left\langle v_R^2\right\rangle-\left\langle v_\phi^2\right\rangle\right)+n\frac{\partial\psi}{\partial R}=0$$

- If we recall the circular velocity $V_c=\sqrt{R\frac{\partial\psi}{\partial R}}$, we get

$$\left\langle v_R^2\right\rangle\frac{R}{n}\frac{\partial n}{\partial R}+\left(\left\langle v_R^2\right\rangle-\left\langle v_\phi^2\right\rangle\right)+V_c^2=0$$

- If we write this as

$$V_c^2-\left\langle v_\phi^2\right\rangle=-\left\langle v_R^2\right\rangle-\frac{R}{n}\frac{\partial}{\partial R}\left(n\left\langle v_R^2\right\rangle\right)$$

- It is reminiscent of an accretion disc with pressure support:

$$v_{\mathrm{Kep}}^2-v_\phi^2=-\frac{R}{\rho}\frac{\partial p}{\partial R}$$

Asymmetric Drift

$$\left\langle v_R^2 \right\rangle \frac{R}{n}\frac{\partial n}{\partial R} + \left(\left\langle v_R^2 \right\rangle - \left\langle v_\phi^2 \right\rangle\right) + V_c^2 = 0$$

- Now we write $v_R = -u$ and $v_\phi = v$. Then
  - $\left\langle v_R^2 \right\rangle = \left\langle v_R \right\rangle^2 + \sigma_u^2 = \sigma_u^2$
  - $\left\langle v_\phi^2 \right\rangle = \langle v \rangle^2 + \sigma_v^2$
- Thus

$$V_c^2 - \langle v \rangle^2 = \sigma_u^2 \left(-\frac{\partial \ln n}{\partial \ln R} + \frac{\sigma_v^2}{\sigma_u^2} - 1 \right)$$

- If the motion is nearly circular, $\langle v \rangle \approx V_c$,

$$V_c - \langle v \rangle = \frac{\sigma_u^2}{2V_c} \left(-\frac{\partial \ln n}{\partial \ln R} + \frac{\sigma_v^2}{\sigma_u^2} - 1 \right)$$

Asymmetric Drift

$$V_c - \langle v \rangle = \frac{\sigma_u^2}{2V_c}\left(-\frac{\partial \ln n}{\partial \ln R} + \frac{\sigma_v^2}{\sigma_u^2} - 1 \right)$$

- Now from epicyclic theory, $\frac{\sigma_v^2}{\sigma_u^2} - 1 \approx -0.5$ (Chpater 6 of booklet)
- But the number density decreases steeply (exponential), so the term in brackets is positive
- Thus the mean velocity $\langle v \rangle < V_c$; this is the *asymmetric drift*
- The larger the velocity dispersion, the greater the lag
- One can obtain the LSR velocity relative to the Solar velocity by extrapolating to $\sigma_u^2 = 0$