Convergence properties of AGIS

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Abstract. Some aspects of the convergence of AGIS (Astrometric Global Iterative Solution) are discussed from a theoretical viewpoint. It is shown that the observed decline of the convergence rate can be understood from general principles. A simple scheme is suggested to accelerate the convergence. However, more efficient ways are likely to be found after a more thorough study of the problem.

1 Introduction

The convergence of AGIS is monitored by looking at the statistical size of updates in successive iterations [3]. The dashed curve and open circles in Fig. 1 shows a typical sequence of updates. A characteristics (and at first unexpected and puzzling) feature of this curve is that the initial rapid convergence later turns into a very slow convergence. Since we are dealing with simulations, we can also study the development of the actual errors, which shows a similar behaviour (solid curve and filled circles in Fig. 1). A qualitatively similar behaviour was observed for the ODIS [2], although much more extreme in terms of the slowness of convergence.

The purpose of this note is to give a qualitative explanation for this as a basis for possible remedies in terms of various acceleration schemes.

There is a vast literature about iterative solutions of large linear systems of equations. All the results given below are probably well-known in the field since many years. The classical reference books on the subject appear to be [4] and [1].

2 Formulation

We are trying to solve a large least-squares problem by the method of normal equations. The resulting system is

\[ Ax = b \]  

where \( A \) is a symmetric, positive semi-definite matrix. (It is semi-definite because, e.g., the system of astrometric positions and proper motions has six degrees of freedom. Apparently the circumstance that \( A \) is not positive-definite has no effect on the present analysis except that we must remember that \( A^{-1} \) does not exist.) The desired (minimum-norm) solution is formally given by

\[ x = A^+ b \]

where \( A^+ \) is the pseudo-inverse of \( A \). In the following we assume that the computation implicit in (1), i.e., pre-multiplying a given vector with the matrix \( A \), is relatively easy
and can be done to full numerical accuracy, while the computation implicit in (2), i.e., the direct (pseudo-) solution of the system, is prohibitively difficult.

Since we cannot compute (2) directly, we choose instead to make a different, but much easier computation, namely

$$\mathbf{x}^{(1)} = B \mathbf{b}$$

(3)

where $B$ is, in some sense, an ‘approximation’ to $A^+$. Recognizing that $\mathbf{x}^{(1)}$ is not exactly what we wanted, we insert it back into (1) and find the (non-zero) residuals

$$r^{(1)} = \mathbf{b} - A \mathbf{x}^{(1)}$$

(4)

We now look for an update $\mathbf{\delta}$ such that $\mathbf{x}^{(1)} + \mathbf{\delta}$ is closer to the desired solution. Inserting $\mathbf{x} = \mathbf{x}^{(1)} + \mathbf{\delta}$ in (1) we find that the update should ideally be the solution of

$$A \mathbf{\delta} = r^{(1)}$$

(5)

Following the same logic as before, we compute an approximate update as

$$\mathbf{\delta}^{(1)} = B r^{(1)}$$

(6)

and consequently obtain

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \mathbf{\delta}^{(1)}$$

(7)
Repeating the process gives a sequence of approximations \( x(1), x(2), x(3), \ldots \) which we hope will converge to \( x \).

The general step from iteration \( k \) to \( k + 1 \) can be summarized as follows:

\[
\begin{align*}
\mathbf{r}(k) &= \mathbf{b} - A\mathbf{x}(k) \\
\mathbf{\delta}(k) &= \mathbf{B}\mathbf{r}(k) \\
\mathbf{x}(k+1) &= \mathbf{x}(k) + \mathbf{\delta}(k)
\end{align*}
\]

We note that the process can be initiated with \( \mathbf{x}(0) = \mathbf{0} \).

### 3 The errors

In the \( k \)th iteration the error vector is

\[
\mathbf{e}(k) = \mathbf{x}(k) - \mathbf{x}
\]

We obtain a recursion formula as follows:

\[
\mathbf{e}(k+1) = \mathbf{x}(k+1) - \mathbf{x} \\
= \mathbf{e}(k) + \mathbf{\delta}(k) \\
= \mathbf{e}(k) + \mathbf{B}(\mathbf{b} - A\mathbf{x}(k)) \\
= \mathbf{e}(k) - \mathbf{BA}\mathbf{e}(k) \\
= (\mathbf{I} - \mathbf{BA})\mathbf{e}(k) \\
= \mathbf{Me}(k)
\]

where \( \mathbf{M} = \mathbf{I} - \mathbf{BA} \) is called the iteration matrix. Since \( \mathbf{e}(0) = -\mathbf{x} \) we find

\[
\mathbf{e}(k) = -\mathbf{M}^k\mathbf{x}
\]

The convergence behaviour is thus entirely defined by the matrix \( \mathbf{M} \). It incorporates both the properties of the least-squares problem itself \( (\mathbf{A}) \) and of the approximation scheme \( (\mathbf{B}) \) adopted for its solution.

### 4 The updates

A corresponding recursion holds for the updates:

\[
\mathbf{\delta}(k+1) = \mathbf{B}\mathbf{r}(k+1) \\
= \mathbf{B}(\mathbf{b} - A\mathbf{x}(k+1)) \\
= \mathbf{B}\left(\mathbf{b} - A(\mathbf{x}(k) + \mathbf{\delta}(k))\right) \\
= \mathbf{B}(\mathbf{b} - A\mathbf{x}(k)) - \mathbf{BA}\mathbf{\delta}(k) \\
= \mathbf{\delta}(k) - \mathbf{BA}\mathbf{\delta}(k) \\
= \mathbf{M}\mathbf{\delta}(k)
\]

Since \( \mathbf{\delta}(0) = \mathbf{Bb} = \mathbf{BAx} \) we find

\[
\mathbf{\delta}(k) = \mathbf{M}^k\mathbf{BAx}
\]
5 Properties of the iteration matrix

For a real symmetric matrix (such as $A$) it is well known that all the eigenvalues are real and that there is an orthogonal set of eigenvectors. Unfortunately, $M$ is non-symmetric, and one cannot in general be sure that the eigenvalues are real. However, in representative cases (e.g. when $B$ is also symmetric), this appears to be the case. For simplicity of the exposition, the eigenvalues are in the following assumed to be real and non-negative, and denoted $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. (Since $A$ is singular, the last few eigenvalues are zero.) The corresponding eigenvectors $u_i, i = 1 \ldots n$ are linearly independent and assumed to be normalized, but they need not be orthogonal.

For any eigenvalue–eigenvector pair $(\lambda_i, u_i)$ we have

$$Mu_i = \lambda_i u_i$$

Repeated application gives

$$M^k u_i = \lambda_i^k u_i$$

From (13) it is seen that $x^{(k)} \to x$ if $M^k$ vanishes as $k \to \infty$. According to (17) this will happen if all the eigenvalues are absolutely less than 1. With $\rho(M)$ denoting the spectral radius of the iteration matrix, the condition for convergence is $\rho(M) < 1$. This is a central result in the area of iterative methods.

6 Convergence properties

Since the eigenvectors are linearly independent they can be used as a basis for the solution vector,

$$x = \sum_{i=1}^{n} c_i u_i$$

We then find by means of (12) and (17)

$$e^{(k)} = -\sum_{i=1}^{n} \lambda_i^k c_i u_i$$

Again we see that $e^{(k)} \to 0$ if the eigenvalues are absolutely less than 1. Since $\lambda_1$ is the largest eigenvalue, this is equivalent to $\lambda_1 < 1$.

The updates can similarly be written in terms of the coefficients $c_i$. Since $M = I - BA$ it follows that the eigenvalues of $BA$ are $1 - \lambda_i$ while the eigenvectors are the same as for $M$. From (15) we therefore have

$$\delta^{(k)} = \sum_{i=1}^{n} (1 - \lambda_i) \lambda_i^k c_i u_i$$

The manner in which $e^{(k)}$ and $\delta^{(k)}$ converge to zero clearly depends on the spectrum of eigenvalues and the relative sizes $c_i$ of the spectral components of $x$. For example, if all the eigenvalues are of the same size $\lambda$, then $e^{(k)} = \lambda^k e^{(0)}$ and the errors (and updates) will decrease in strictly geometric progression. In reality the eigenvalues are different, and the convergence will not be geometric.
Figure 2: The expected convergence behaviour for a problem with two orthogonal error components of amplitude 0.01 and 1, corresponding to the eigenvalues 0.9 and 0.2 of the iteration matrix.

To illustrate this, let us assume that the spectral decomposition of the solution vector in (18) contains just two non-zero terms, viz. for indices $i$ and $j$ (with $i < j$). Assume that the corresponding eigenvalues are (say) $\lambda_i = 0.9$ and $\lambda_j = 0.2$. The first component ($i$) thus represents a pattern of errors (in our initial estimate of $x$) that is relatively difficult to reduce by the chosen iteration method; it could typically be some large-scale systematics in the astrometric parameters. The second component ($j$) represents an error pattern that is reduced to one fifth of its previous amplitude with each iteration. This could be the case for the high spatial-frequency part of the initial astrometric errors: in AGIS, such errors are efficiently damped by averaging over the astrometric FOV before they are injected into the attitude. It is reasonable to assume that the initial high-frequency errors have a much larger amplitude than the large-scale systematics, so that $|c_i| \ll |c_j|$. Let us assume that the amplitudes of the components in some suitable unit are $c_i = 0.01$ and $c_j = 1$, and that the corresponding eigenvectors are orthogonal, $u_i^T u_j = 0$ (in the suggested scenario this would very nearly be the case). With these assumptions we have $|e^{(k)}| = [(0.9^k \cdot 0.01)^2 + (0.2^k \cdot 1)^2]^{1/2}$ and $|\delta^{(k)}| = [(0.1 \cdot 0.9^k \cdot 0.01)^2 + (0.8 \cdot 0.2^k \cdot 1)^2]^{1/2}$, shown in Fig. 2 for the first ten iterations. In the first few iterations, the second component dominates, by virtue of its larger amplitude, but since this component is rapidly damped, subsequent iterations become dominated by the more slowly decreasing first component. This gives the change of slope seen in the figure. A more gradual slowing of the iteration rate could be obtained by including more terms in the model.

The qualitative similarity between Figs. 1 and 2 suggests that we can understand the slowing-down of the AGIS convergence rate as a gradual shift of the dominating error components from large-amplitude, small-scale errors to more slowly-damped components.

From (19) and (20) it is clear that, after a sufficient number of iterations, the first term
will dominate in each sum, so that the error and update decrease geometrically:

\[
e^{(k)} \simeq -\lambda_1^k u_1
\]

(21)

\[
\delta^{(k)} \simeq (1 - \lambda_1)\lambda_1^k u_1
\]

(22)

In Fig. 1 this appears to happen after the first 35 or so iterations. At this point we can estimate \(\lambda_1\) as the ratio of successive updates:

\[
\lambda_1 \simeq \frac{\left| \delta^{(k)} \right|}{\left| \delta^{(k-1)} \right|}
\]

(23)

and use (21)–(22) to predict \(e^{(k)}\). It should then be possible to eliminate most of the dominating error component by applying, instead of \(\delta^{(k)}\), the single update

\[
\delta = -e^{(k)} \simeq (1 - \lambda_1)^{-1}\delta^{(k)}
\]

(24)

If this is followed by a few more ‘regular’ iterations, using (8), it should be possible to identify the second largest eigenvalue of \(M\), and the corresponding error component, which could then be eliminated in analogy with (24).

It should be noted that various modifications of \(A\) and/or \(B\), which may superficially seem trivial (such as preconditioning the equations or changing the order of iterations) can change the spectrum of \(M\) and thus potentially lead to improved convergence properties.

7 Conclusions

The general behaviour of the AGIS convergence, as exemplified by Fig. 1, can be understood as a simple and natural consequence of the intrinsic properties of the adopted iteration scheme. This insight can be used to accelerate the solution as described above, but more efficient schemes can perhaps be found through a deeper study of the problem.

References


Acronyms

The following table has been generated from the on-line Gaia acronym list:

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
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<tbody>
<tr>
<td>AGIS</td>
<td>Astrometric Global Iterative Solution</td>
</tr>
<tr>
<td>FOV</td>
<td>Field of View (also denoted FOV)</td>
</tr>
<tr>
<td>ODIS</td>
<td>One-Day Iterative Solution</td>
</tr>
</tbody>
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