1 Introduction

Requirements on the along-scan attitude (pointing) noise for the on-ground attitude determination (OGAD) were derived in SAG–LL–020 (26 June 1998). The present document is a re-discussion and re-formulation of the noise requirements based on current Gaia-2 parameters and more stringent considerations. It is concerned only with the along-scan pointing error (which is the stricter requirement), and only with frequencies below $\sim 0.3$ Hz (corresponding to the CCD integration).

The currently foreseen OGAD (e.g., as implemented in the GDAAS study) consists of a robust, weighted least-squares fitting of cubic spline functions to positional observations (centroid location in the CCD data stream) collected in ASM and AF1–11. The spline functions describe the four components of the attitude quaternion as smooth functions of time. This attitude determination is part of the Global Iterative Solution for the attitude, instrument parameters and astrometric parameters of a subset of well-behaved sources. In the final iteration the attitude determination is unaffected by initial errors of the instrument and source parameters. It can be assumed that the residual errors in the calibration and source parameters contribute negligibly to the OGAD attitude error.

2 Proposed requirement on the AL attitude noise

2.1 Primary requirement

The primary requirement on the along-scan attitude noise (below $\sim 0.3$ Hz) is that its contribution, in a quadratic sense, to the parallax error of a bright source is less than 1 $\mu$as RMS (mean sky conditions).

The requirement is derived from the need to reach very high astrometric accuracy for bright sources, important (among other things) in planet detection. It is envisaged that
there will be an ‘accuracy floor’ of order 4 \( \mu \text{as} \) for bright sources, as set by CCD saturation, calibration errors, thermo-mechanical instabilities and OGAD errors. Since the contributions from many of these error sources are rather uncertain, and the OGAD contribution is only one of many foreseen error sources, the requirement must be set a level significantly below this floor.

### 2.2 Derived requirement

While attitude errors may be correlated on time scales comparable with a FOV transit (of order 40 s), it is reasonable to assume that the OGAD errors on successive FOV transits of a given source are uncorrelated. Thus, the mean attitude error on a single FOV transit is attenuated by the same coefficient of improvement as applicable to the photon noise per FOV transit. With current Gaia-2 parameters the relevant coefficient of improvement is

\[
g(\pi) N_{\text{transit}}^{-1/2} = 0.212,
\]

where \( g(\pi) = 1.93 \) is the mean geometrical parallax factor resulting from the NSL with 50° solar aspect angle, and \( N_{\text{transit}} = 83 \) is the mean number of AF transits per source (GAIA–JdB–008, Table 1). Thus, at the level of an individual FOV transit the primary requirement translates to

\[
\sigma_{\text{FOV}} \leq 4.7 \mu\text{as}
\]  

### 2.3 Further interpretation

The quantity \( \sigma_{\text{FOV}} \) should be understood in the following way.

Let \( a(t) \) be the along-scan attitude as function of time (e.g. the corresponding Euler angle w.r.t. the nominal attitude) and let

\[
\bar{a}_{h}(t) = \frac{1}{h} \int_{t-h/2}^{t+h/2} a(t') \, dt'
\]  

be the moving average of \( a(t) \) with box length \( h \). For the OGAD there are (at least) three important time scales to consider, each associated with an averaging over some interval \( h \): the CCD integration time \( \tau = 3.3 \) s, the spline knot interval \( \Delta t \) (its value to be considered below) and the FOV transit duration \( T = 40 \) s.

It can be noted that \( a(t) \) is actually not observable in the AF, because all observations are averaged over the CCD integration time \( \tau \) (cf. Bastian, ref?). Thus, the function actually observed and determined by the OGAD is \( \bar{a}_{\tau}(t) \). Special requirements apply to the difference \( a(t) - \bar{a}_{\tau}(t) \) as part of the relative pointing error, but they are not considered here; in fact the difference between \( a(t) \) and \( \bar{a}_{\tau}(t) \) is ignored in the following.

Let \( s(t) \) be the spline function fitted to \( a(t) \) as part of the OGAD and let

\[
e(t) = a(t) - s(t)
\]  

be the OGAD error. Inequality (1) should be understood as a requirement on the mean OGAD attitude error over any time interval of length \( T = 40 \) s, the duration of a transit across AF1–11; thus

\[
\sigma_{\text{FOV}} = \langle \bar{e}_{T}(t)^{2} \rangle^{1/2}
\]
where the angular brackets denote a statistical average.

3 Evaluation in the frequency domain

In this section the previous requirement is translated into a tentative limit on the spectral density of the FEEP noise. To this end the total OGAD error $e(t)$ is supposed to consist of two components,

$$e(t) = o(t) + m(t)$$

where $o(t)$ is the low-frequency part dominated by observational noise in the AF centroiding data, and $m(t)$ is the high-frequency part consisting of modelisation errors. The separation frequency is essentially determined by the interval $\Delta t$ between successive spline knots (assumed to be constant). Attitude noise above the spline separation frequency $f_s \approx 1/2\Delta t$ cannot be modelled by the spline and are therefore transmitted without attenuation to the OGAD error. Lower-frequency noise is effectively absorbed by the spline, and thus strongly attenuated. However, even if the attenuation is complete, there is still a low-frequency OGAD component resulting from the observational noise in the AF data.

An important assumption is that the separation frequency is much higher than the attitude control bandwidth (which may be of order 1 mHz). This means that attitude errors caused by the star tracker and real-time rate measurements are perfectly absorbed by the spline and therefore do not contribute at all to the OGAD error. This assumption is almost certainly valid if $\Delta t < 100$ s.

3.1 Frequency characteristics of the spline

The filtering characteristics of a cubic spline (defined on an equidistant $\Delta t$ knot sequence) have been evaluated through numerical experiments. The purely empirical results are shown in Figures 1 and 2.

Figure 1 shows the attenuation factor $K$ (in power, i.e. the square of the amplitude) resulting from fitting a spline to a harmonic oscillation of frequency $f$, as function of the dimensionless number $f\Delta t$. It is seen that oscillations at $f\Delta t < 0.2$ are very efficiently damped (i.e., absorbed by the spline), while those at $f\Delta t > 0.7$ are not at all damped. In the critical transition region the following approximation may be used:

$$K_{\Delta t}(f) = \frac{1}{1+(2f\Delta t)^{-12}}$$

Figure 2 shows the power spectral density of a cubic spline fitted to white-noise data, again plotted against $f\Delta t$. Again, the transition is extremely sharp and defines an effective bandwidth of $0.475/\Delta t$. The curve in the figure can be approximated by a step function:

$$S_{\Delta t}(f) = \begin{cases} 1, & f < 0.475/\Delta t \\ 0, & \text{otherwise} \end{cases}$$
Figure 1: Filter characteristics for a cubic spline defined on an infinite, equidistant knot sequence with knot separation $\Delta t$. The abscissa is the normalised frequency $f \times \Delta t$. The ordinate is the residual power, i.e. the fraction of power in the residuals of a spline fit to a harmonic oscillation with frequency $f$. A residual power of $10^{-4}$ means that the amplitude of the residuals is $10^{-2}$ smaller than the amplitude of the fitted oscillation.

The solid curve is the approximation $\left[1 + \left(\frac{2f}{\Delta t}\right)^{-12}\right]^{-1}$ valid for $f\Delta t > 0.15$, the dashed curve shows the approximation $\left[1 + 0.5(f\Delta t)^{-8}\right]^{-1}$ valid for lower frequencies.

3.2 Power spectral density of the OGAD error

The power spectral density (PSD) of $e(t)$ is\(^1\)

$$P_e(f) = P_o(f) + P_m(f)$$

(8)

where $P_o(f)$ and $P_m(f)$ are the PSD of $o(t)$ and $m(t)$. If $P_o(f)$ represents the high-frequency part of the attitude noise (at least for $f > 1/2T \simeq 10^{-2}$ Hz), then we have $P_m(f) = P_o(f)K_{\Delta t}(f)$. Similarly, if $A$ is the low-frequency PSD of the AF data (assumed to be white noise), then $P_o(f) = AS_{\Delta t}(f)$. Thus,

$$P_e(f) = AS_{\Delta t}(f) + P_o(f)K_{\Delta t}(f)$$

(9)

The FOV averaging corresponds in the PSD domain to the application of the filter function $\text{sinc}^2(\pi f T)$. We have, therefore,

$$\sigma_{\text{FOV}}^2 = \int_0^\infty P_{\tilde{e}}(f) \, df = \int_0^\infty \left[AS_{\Delta t}(f) + P_o(f)K_{\Delta t}(f)\right] \text{sinc}^2(\pi f T) \, df$$

(10)

\(^1\)The following normalisation convention is used for the PSD of $x(t)$: $\int_0^\infty P_x(f) \, df = \sigma_x^2$. 

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Writing

\[ \sigma_{\bar{o}}^2 = \int_0^{\infty} AS_{\Delta t}(f) \text{sinc}^2(\pi f T) \, df, \quad \sigma_{\bar{m}}^2 = \int_0^{\infty} P_a(f) K_{\Delta t}(f) \text{sinc}^2(\pi f T) \, df \]  

we have

\[ \sigma_{\text{FOV}}^2 = \sigma_{\bar{o}}^2 + \sigma_{\bar{m}}^2 \]  

It remains to estimate \( A \) and \( P_a(f) \).

### 3.3 Estimation of the low-frequency observation noise

In order to estimate \( A \), the low-frequency PSD of the observation noise, I assume that at least half of all stars brighter than magnitude \( G = 15 \) can be used for the OGAD. The density of such stars varies roughly by a factor 10 between the galactic equator \( (b = 0^\circ) \) and poles \( (b = \pm 90^\circ) \). The attitude noise requirement must be based on the least favourable configuration that is regularly encountered in normal operations. In SAG–LL–020 the density at the galactic poles was assumed. However, for the along-scan attitude, it is the combined density in the two astrometric fields that matter, and since the fields are nearly at right angle to each other, it will never happen that both point to the galactic pole at
the same time. The lowest combined density will rather happen when both fields are at an intermediate latitude ($b \simeq 45^\circ$). In the following I therefore assume the average star density for $|b| > 30^\circ$. In this region, the density $N(G)$ of stars brighter than magnitude $G$, per square degree on the sky, is given by:

$$\log N(G) = 2.47 + 0.287(G - 15) - 0.014(G - 15)^2$$  \hspace{1cm} (13)

The relation holds for $12 \leq G \leq 20$. Random numbers $G$ following this distribution with the limit $G < G_{\text{max}}$ can be generated as

$$G = 25.25 - \left[105.0625 - 20.5(G_{\text{max}} - 15) + (G_{\text{max}} - 15)^2 - 31.021 \ln R\right]^{1/2}$$  \hspace{1cm} (14)

where $R$ is a uniform random deviate in $(0,1)$.

It is assumed that half of the stars up to $G_{\text{max}} = 15$ can be used for the attitude determination, or $148 \text{ deg}^{-2}$. The sky area swept by detectors AF1–11 (FOV $0.65^\circ$ across scan) at the scan rate of $60^\circ$ per hour is $2 \times 11 \times 0.65 \times 60 = 858 \text{ deg}^2 \text{ hr}^{-1}$, resulting in 35 observations per second of such stars.

The along-scan centroiding accuracy (per observation) has been evaluated as part of the provisional accuracy analysis for Gaia-2 (assuming Ag coatings, CCD#1B, and spectral type G2V). In the magnitude interval $12 \leq G \leq 20$ the result (including a 20% margin) is

$$\sigma_\xi^2 = 2700 + 530u + 0.05u^2 \text{ [\muas]}$$  \hspace{1cm} (15)

where $u = 10^{0.4(G-10)}$. For $G = 15$ we have $\sigma_\xi = 237 \text{ \muas}$.

Monte Carlo experiments were performed, in which elementary observations with zero mean value and standard deviation $\sigma_\xi$ were simulated at random points in time over a 3-hour interval. The mean rate of observations and their magnitude distribution (up to $G_{\text{max}} = 15$) followed the prescriptions above. A cubic spline was then fitted to the data, using weighted least squares. Finally, the RMS value of the spline ($\sigma_s$) was calculated and the experiment repeated several times using different knot intervals. Assuming that the PSD of the spline is $A S_{\Delta t}(f)$, we expect to find a constant value $A = \sigma_s^2 \Delta t/0.475$ independent of $\Delta t$. In fact $A$ was found to vary only by a few per cent for $\Delta t = 5 \text{ to } 20 \text{ s}$, with a mean value of $A = 860 \text{ \muas}^2 \text{ Hz}^{-1}$.

From (11) we find (with $T = 40 \text{ s}$) that $\sigma_\xi = (A/2T)^{1/2} = 3.28 \text{ \muas}$ in the limit when $\Delta t \ll T$. For $\Delta t = 5, 10, 20, 40, 80 \text{ s}$ we find $\sigma_\xi = 3.24, 3.20, 3.12, 2.84, 2.19 \text{ \muas}$. Thus, the requirement (1) is satisfied for any $\Delta t$ if

$$\sigma_m < 3.4 \text{ \muas}$$  \hspace{1cm} (16)

### 3.4 Contribution from FEEP noise

The OGAD is sensitive to FEEP noise mainly at frequencies above $1/2\Delta t$. Assuming

1. that the torque noise produced by the FEEPs (and other perturbations if relevant) can be considered as white noise at these frequencies, with a (constant) PSD of $P_{\text{torque}} \left[(\text{Nm})^2\text{Hz}^{-1}\right]$, and
2. that the satellite behaves as a rigid body with moment of inertia $I_{XX}$ about the spin axis, with negligible cross-terms of the inertia tensor, then
\[ P_a(f) = Q^2 (2\pi f)^{-4} \] (17)
where $Q = P_{\text{torque}}^{1/2}/I_{XX}$ is the angular acceleration noise in $[\text{rad s}^{-2} \text{ Hz}^{-1/2}]$.

In the integral (11) for $\sigma_m^2$ we can approximate $K_{\Delta t}(f)$ with 1 for $f > 1/2\Delta t$ and with $(2f\Delta t)^{12}$ for lower frequencies. Similarly the factor $\text{sinc}^2(\pi f T)$ can be approximated by 1 for $f < 1/4T$ and with $(4fT)^{-2}$ for higher frequencies (this choice of break frequency preserves the correct integral $1/2T$ of the $\text{sinc}^2$ function). With (17), and assuming that $\Delta t < 2T$, the result is
\[ \sigma_m \simeq 4.33 \times 10^9 Q \Delta t^{5/2} T^{-1} \] (18)
when expressed in $\mu$as.

The previously derived limit $\sigma_m < 3.4 \mu$as then leads to the following tentative requirement on the FEEP angular acceleration noise in the vicinity of the frequency $1/2\Delta t$:
\[ Q < (3.14 \times 10^{-8} \text{ rad s}) \Delta t^{-5/2} \] (19)
It is seen that the observation noise per FOV transit, $\sigma_o$, is rather insensitive to the choice of spline knot interval $\Delta t$, as long as it is less than $T = 40$ s. On the other hand, the modelisation error $\sigma_m$ decreases as the power 2.5 of the knot interval. Thus, to minimise the total error $\sigma_{\text{FOV}}$ we should try to make $\Delta t$ as small as possible. It is not clear what the minimum feasible value is (from the point of view of OGAD implementation, micrometeoroid detection, etc.), but a reasonable guess is a knot interval somewhat larger than the CCD crossing time, or (say) $\Delta t = 5$ s. Provisionally adopting this value (a factor 2 smaller than previously considered) results in the limit
\[ Q < 5.6 \times 10^{-10} \text{ rad s}^{-2} \text{ Hz}^{-1/2} \] (20)
for the angular acceleration noise around 0.1 Hz.

### 4 Conclusions

A high-level requirement on the on-ground attitude determination (OGAD) accuracy is proposed, Eq. (1), which could be tested against a detailed attitude simulation including known perturbations, the control loop, observation noise, and the OGAD procedure.

Under specific assumptions (which should be verified through simulation), the high-level requirement is translated into a requirement on the OGAD modelisation error, Eq. (16). With additional assumptions (which again should be verified) this is in turn translated

\[ \text{Direct numerical integration of the second integral in (11), with } T = 40 \text{ s and } \Delta t = 5 \text{ s, results in the limit } Q < 7.0 \times 10^{-10} \text{ rad s}^{-2} \text{ Hz}^{-1/2}. \] (25% difference between the numerically calculated value and (20) is consistent with the kind of approximations discussed above. (I am grateful to Mr. César García Marirrodriga, ESTEC, for helping me to correct a serious error in the original version of this document.)
into a requirement on the angular acceleration noise around 0.1 Hz, Eq. (20), of order
$6 \times 10^{-10} \text{ rad s}^{-2} \text{ Hz}^{-1/2}$.

For a typical Gaia spacecraft moment of inertia, $I_{XX} \sim 3200 \text{ kg m}^2$, and $\sim 1 \text{ m lever arm}$, the corresponding limit on the FEEP thrust noise is of order $2 \mu\text{N Hz}^{-1/2}$. 