1 Introduction

This note describes the calculation of the two-dimensional polychromatic Point Spread Function (PSF) for GAIA, including realistic aberrations and the effects of pixel binning, TDI smearing and transverse motion.

The polychromatic PSF depends on the spectral distribution of the object, folded with the detector quantum efficiency (QE). Rather than computing the PSF for different combinations of spectral type, interstellar reddening and QE curves, a rough parametrisation of the effective spectrum is proposed, using a linear spline approximation. This allows separation of the PSF calculation into just four spectral components.

2 Approximation of spectra by linear splines

In the following I use effective spectrum to mean the product of the incident photon flux spectrum, instrument transmittance and detector quantum efficiency: \( E(\lambda) = f_\lambda T_\lambda Q_\lambda \).

For a given effective spectrum the polychromatic PSF is obtained as a weighted sum of the monochromatic PSFs. For many purposes, such as studies of object detection and accuracy analysis, there is however no need to use a very detailed spectrum for the PSF calculation. (An exception is chromaticity studies, cf. SAG–LL–024.) A coarse characterisation of \( E(\lambda) \) by a few parameters should then suffice. These parameters could represent the strengths of certain spectral components, or basic spectra.

Assume that the effective spectrum is approximated by the function \( s(\lambda) \) such that

\[
E(\lambda) \sim s(\lambda) = \sum_{k=1}^{n} s_k B_k(\lambda) ,
\]

where \( B_k(\lambda) \) are suitable (non-negative) basic spectra. Then the PSF for the effective spectrum is approximately given by

\[
P(\xi, \eta) = \sum_{k=1}^{n} s_k P_k(\xi, \eta) ,
\]

where \( P_k(\xi, \eta) \) is the PSF for the spectral distribution \( B_k(\lambda) \). The advantage of this formulation is that the PSF only needs to be computed once for each basic spectrum, rather than for every possible spectrum. If \( n \) is small enough, this is really a big advantage.

In the following I assume that \( s(\lambda) \) is continuous and piecewise linear.
Table 1. Coefficients $s_k$ of the basic spectra in Fig. 1 as functions of $V - I$ for the quantum efficiency curves of two different CCDs.

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Mathematically, the basic spectra are the basis functions (also called basic splines) of the linear spline function $s(\lambda)$ defined on the know sequence $\lambda_0, \lambda_1, \ldots, \lambda_{n+1}$, with end conditions $s(\lambda_0) = s(\lambda_{n+1}) = 0$. The knot sequence uniquely defines the basic spectra

$$B_k(\lambda) = \frac{2}{\lambda_{k+1} - \lambda_k} \times \begin{cases} 0 & \text{if } \lambda < \lambda_{k-1} \\ (\lambda - \lambda_{k-1})/(\lambda_k - \lambda_{k-1}) & \text{if } \lambda_{k-1} \leq \lambda < \lambda_k \\ (\lambda_{k+1} - \lambda)/(\lambda_{k+1} - \lambda_k) & \text{if } \lambda_k \leq \lambda < \lambda_{k+1} \\ 0 & \text{if } \lambda_k \leq \lambda \end{cases} \quad (3)$$

for $k = 1 \ldots n$. The first factor normalises the integral of each basic spectrum.

Since we are here only interested in the shape of the effective spectrum, $s(\lambda)$ may be normalised to unit area. Since the basic spectra are also normalised it follows that

$$\sum_{k=1}^{n} s_k = 1. \quad (4)$$

A linear spline approximation with the $n = 4$ basic spectra in Fig. 1 may be adequate for many purposes. Figs. 2–6 show how well some different spectra can be approximated by these functions. The knot sequence used here is $\lambda_{0,5} = 300, 370, 460, 650, 850$ and $1050$ nm. The end knots are set by the assumed QE curves being strictly zero below $300$ nm and above $1050$ nm. Interior knots were selected to represent major ‘breakpoints’ in real spectra, e.g. $370$ nm near the Balmer jump. The spectral shape is then characterised by the four spline coefficients $s_1$ to $s_4$.

These spline coefficients were computed for a variety of effective spectra, using the 175 stellar spectra from Gunn & Stryker, plus interstellar reddening ranging from 0 to 6 mag, and the two standard CCD quantum efficiency curves (CCD#1 for Spectro, CCD#3 for Astro). For each effective spectrum the colour index $V - I$ was also computed as described in SAG–LL–017. Figures 7 and 8 show the resulting spline coefficients as function of $V - I$. As expected, the relation is not unique, but to within about $\pm 0.03$ the coefficients can be predicted from $V - I$. Table 1 gives the approximate coefficients as function of $V - I$.

For spectra with $V - I > 4.5$ to 5 mag the approximations become questionable, as most of the flux falls longward of $850$ nm. Using Table 1 and Eqs. (1) and (3) a reasonable approximation of the effective spectrum can be obtained for any $V - I < 5$. It is then sufficient to compute the PSF for the four basic spectra in Fig. 1.
Figure 1. Definition of four basic spectra $B_k(\lambda)$, $k = 1 \ldots 4$, of triangular shape. In Figs. 2–6 it is shown how various effective spectra can be approximated by linear combinations of these functions.

Figure 2. The photon flux spectrum of an unreddened O5 star, multiplied by the quantum efficiency of CCD#3 (Astro), and its approximation in terms of the four basic spectra in Fig. 1. The vertical scale is arbitrary.
Figure 3. The photon flux spectrum of an unreddened A1V star, multiplied by the quantum efficiency of CCD#3 (Astro), and its approximation in terms of the four basic spectra in Fig. 1. The vertical scale is arbitrary.

Figure 4. The photon flux spectrum of a unreddened G2V star, multiplied by the quantum efficiency of CCD#3 (Astro), and its approximation in terms of the four basic spectra in Fig. 1. The vertical scale is arbitrary.
Figure 5. The photon flux spectrum of an unreddened M8V star, multiplied by the quantum efficiency of CCD#3 (Astro), and its approximation in terms of the four basic spectra in Fig. 1. The vertical scale is arbitrary.

Figure 6. The photon flux spectrum of a K3III star, reddened to $A_V = 2$ mag, multiplied by the quantum efficiency of CCD#3 (Astro), and its approximation in terms of the four basic spectra in Fig. 1. The vertical scale is arbitrary.
Figure 7. The coefficients $s_1$ to $s_4$ calculated for a number of unreddened and reddened stellar spectra, plotted versus the $V - I$ colour index computed from the same spectra. The QE curve of CCD#1 (Spectro) was assumed.

Figure 8. Same as Fig. 7 but for the QE curve of CCD#3 (Astro).
3 Calculation of the monochromatic optical PSF

Let \((x, y)\) be linear coordinates in the pupil plane and \((\xi, \eta)\) angular coordinates in the image plane, with \(x\) in the direction along the scan. For a given wavefront error (WFE) map \(w(x, y)\), valid at a specific point in the field, the monochromatic PSF is given by

\[
P_\lambda(\xi, \eta) = \lambda^{-2} \left| \int_{-H/2}^{H/2} \int_{-D/2}^{D/2} \exp \left[ \frac{2\pi i}{\lambda} (x\xi + y\eta + w(x, y)) \right] \, dx \, dy \right|^2
\]  

(5)

where \(D = 1.7\) m and \(H = 0.7\) m are the dimensions of the pupil along \(x\) and \(y\). This is the optical PSF, caused by diffraction only, and does not yet include the effects of the pixel size and image motion (Section 5).

The practical calculation is best done by means of the Fast Fourier Transform (FFT). This requires that \(x\) and \(\xi\) are discretised into \(N_x\) points, and \(y\) and \(\eta\) into \(N_y\) points. \(N_x\) and \(N_y\) are usually chosen to be powers of 2, which gives the most efficient computation. The choice of \(N_x\) and \(N_y\) and the discretisation steps \(\Delta x, \Delta y, \Delta \xi\) and \(\Delta \eta\) is governed by the basic relations

\[
N_x \Delta x \Delta \xi = N_y \Delta y \Delta \eta = \lambda
\]  

(6)

and the circumstance that \(x, y, \xi\) and \(\eta\) become periodic functions with periods \(N_x \Delta x, N_y \Delta y, N_x \Delta \xi\) and \(N_y \Delta \eta\), respectively. Moreover, it is desirable to make \(\Delta \xi\) and \(\Delta \eta\) independent of wavelength, which means that \(\Delta x\) and \(\Delta y\) must scale in proportion to the wavelength.

I have chosen to use a quarter of a pixel as the step in \(\xi\) and \(\eta\), i.e. \(\Delta \xi = 4.5 \times 10^{-8}\) rad (9.28 mas) and \(\Delta \eta = 13.5 \times 10^{-8}\) rad (27.85 mas). For practical reasons the number of steps was limited to \(N_x = N_y = 1024\). This means that the computed diffraction pattern will have a period of 9.5 arcsec in \(\xi\) and 28.5 arcsec in \(\eta\). The corresponding sampling of the pupil plane varies from \(\Delta x = 6.5\) mm at 300 nm wavelength to 22.8 mm at 1050 nm, and from 2.2 mm to 7.6 mm for \(\Delta y\). This should give a more than adequate sampling of the WFE maps.

The WFE is taken from MMS’s description in 15 points (Fig. 9), given in the form of coefficients for the Zernike polynomials complete to third degree (16 coefficients per point).

The monochromatic PSF is thus obtained as the squared modulus of the FFT of the complex amplitude in the pupil plane, i.e. \(\exp[2\pi i w(x, y) / \lambda] \) inside the pupil \( (|x| < D/2\) and \(|y| < H/2)\) and 0 otherwise. The PSF is normalised such that the sum equals 16 (since the the pixels are oversampled 4 times in each coordinate).

4 PSF for the basic spectra

The PSF for basic spectrum \(B_k(\lambda)\) (Fig. 1) is computed as the weighted mean of the monochromatic PSFs at the six wavelengths

\[
\lambda_j = \lambda_k + (j/6)|\lambda_{k+\text{sign}(j)} - \lambda_k|, \quad j = \pm 1, \pm 3, \pm 5.
\]  

(7)

The corresponding weights are

\[
w_j = \frac{6 - |j|}{9} \times \frac{|\lambda_{k+\text{sign}(j)} - \lambda_k|}{\lambda_{k+1} - \lambda_{k-1}}.
\]  

(8)
5 Pixel binning, TDI smearing and transverse motion

Pixel binning corresponds to a convolution of the optical PSF with a rectangular box function equal to the size of the pixel ($W_\xi \times W_\eta \text{ rad}^2$). Again, this operation is most conveniently carried out by means of the FFT: the optical PSF for the basic spectrum is inverse Fourier transformed to the pupil plane, multiplied by $\text{sinc}(\pi W_\xi x/\lambda)\text{sinc}(\pi W_\eta y/\lambda)$, and then Fourier transformed back to the image plane. Note that $x$ and $y$ scale linearly with $\lambda$, so the arguments of the sinc functions are actually not functions of $\lambda$.

The TDI smearing corresponds to a convolution in the $\xi$ direction with a rectangular function of width $W_\xi/m$, where $m$ is the number of phases per pixel transfer. The GAIA CCDs may use four-phase charge transfer, in which case $m = 4$. However, other effects such as charge diffusion, residual distortion and scan rate errors probably increase the effective width of the convolution function. To be conservative, I assume that the total width of the smearing is a full pixel, i.e. equal to $W_\xi$.

The transverse motion of the image depends mainly on the nominal scanning law, producing a sinusoidal variation of the transverse scan rate with amplitude $\pm 0.17$ arcsec/s. Distortion and rate errors are probably negligible by comparison, if they are about the same as in the along-scan direction. The typical (rms) transverse scan rate is then $0.7 \times 0.17 = 0.12$ arcsec/s. For an integration time of 0.86 s this gives a rectangular smearing of width 0.1 arcsec or 0.9 pixel. To be conservative, I assume again that the typical smearing corresponds to one pixel, i.e. $W_\eta$.

The combined effect of the pixel binning and the assumed smearing in both coordinates is then achieved by means of the spatial filter function $\text{sinc}^2(\pi W_\xi x/\lambda)\text{sinc}^2(\pi W_\eta y/\lambda)$.

6 Shifting and cropping

All the FFT operations described above work on complex data in arrays of $N_x \times N_y = 1024 \times 1024$. The origin in $(\xi, \eta)$ is at the first data point (1,1). Before outputting any results, the origin needs to be shifted to the centre of the array [say, (513,513)].

Most of the outer part of the diffraction image contains very little energy. One can therefore afford to remove most of it and just keep a central portion centred on $\xi = \eta = 0$. I have chosen to keep a little more than half the total width, viz. 517 points in each coordinate (with the 259th point at the orgin). In this way it is possible to extract subsets of the data corresponding to PSFs that are offset by $\pm 0.25$ and $\pm 0.5$ pixel in each coordinate. Each such subset (sampled at one pixel resolution) consists of $129 \times 129$ pixels, or approximately $4.8 \times 14.4$ arcsec$^2$.

The four times oversampling in each coordinate is dense enough to allow accurate interpolation with simple algorithms. This means that a pixel map of $129 \times 129$ points is readily obtained for any offset in the range $\pm 0.5$ pixel.
Figure 9. Definition of field points 1 to 15, for which MMS have provided WFE maps. The gray area is the Astrometric Field (0.56×0.67 deg$^2$). Point 0 designates the case of no aberrations.

7 Data files and output format

Currently available results were calculated for the WFE case ‘Lam50,M2’ defined by MMS ($\lambda$/50 rms on M2 and $\lambda$/30 rms on M1) and consist of 128 PSF files in the PGM format (see below). The following convention was used to name the files:

- Optical PSFs (no pixel binning or smearing): paXXYYY.pgm
- Pixel-binned and smeared PSFs: pbXXYYY.pgm

where XX is the field point (00 through 15; Fig. 9) and YYY the central wavelength (370, 460, 650 or 850; Fig. 1) of the basic spectrum. Figure 10 shows sample data from some of the calculated PSFs.

The single-precision arithmetics restricts the dynamic range of the calculated PSFs to about 7 decades (17 mag). The output takes this into account by storing the integer values nint($10^7P$) in the portable graymap format (PGM files). These are simple text files, using lines of at most 70 characters, and readable with several software packages such as IDL and xv.
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Figure 10. Sample data from the PSF files (7 x 7 pixels centred on the origin). Left: optical PSF, right: pixel-binned and smeared PSF. The indices ix/iy run from 1 to nx/ny, corresponding to steps of a quarter of a pixel. The centre (origin) of the PSF is at ix = iy = 259.
8 Synthesis of polychromatic binned PSF

In this section I discuss briefly the practical use of the PSF files for creating polychromatic PSFs and the special PSFs corresponding to a larger sample than $1 \times 1$ pixel.

The generation of the polychromatic PSF, for samples of $1 \times 1$ pixel and with TDI smearing of $1$ pixel along $\xi$, is just an application of Eq. (2), given the coefficients $s_1$ to $s_4$. In general these coefficients are obtained by fitting the spline function (1) to the effective spectrum, but to a useful approximation they are also given by Table 1 as function of the CCD type and colour index. For instance, the PSF for a $V - I = 1$ star observed with CCD#3 at field point #13 with samples of $1 \times 1$ pixel may be calculated as

$$\text{PSF} = 0.15 \ast \text{pb13}_460 + 0.50 \ast \text{pb13}_650 + 0.35 \ast \text{pb13}_850,$$

where $\text{pb13}_460$ is the PSF represented in the file $\text{pb13}_460$.pgm, etc.

If samples of $k_\xi \times k_\eta$ are used, then the corresponding PSF is obtained by summing the corresponding values in the original PSF. The resulting PSF is still sampled with the same resolution as before, i.e. a quarter of a pixel (or $1/4 k_\xi$ sample in $\xi$ and $1/4 k_\eta$ sample in $\eta$). E.g. for $k_\xi = 2$ and $k_\eta = 3$ the sums could be made as follows:

$$P_{i,j} = \frac{p_{i-2,j-8} + p_{i-2,j-4} + p_{i-2,j} + p_{i-2,j+4} + p_{i-2,j+8} + p_{i+2,j-8} + p_{i+2,j-4} + p_{i+2,j} + p_{i+2,j+4} + p_{i+2,j+8}}{2},$$

where $p$ is the original matrix of pixel values and $P$ the matrix of sample values (with index $i$ along $\xi$ and $j$ along $\eta$). Note that the summing is symmetric in both $\xi$ and $\eta$, leaving the origin of the PSF map unchanged. In practice the binning can be made separately in $\xi$ and $\eta$.

Charge transfer made in steps of $\kappa$ pixels (instead of a single pixel as assumed in Section 5) can be simulated by putting the number of phases in the charge transfer to $m = 1/\kappa$. However, the same result is obtained by averaging $\kappa$ adjacent pixel values. E.g. for $\kappa = 2$:

$$P_{i,j} = \frac{[p_{i-2,j} + p_{i+2,j}]}{2}.$$

The binning (by $k_\xi$ and $k_\eta$) and averaging (by $\kappa$) can be done sequentially in any order, since convolution is an associative operator.

A special program (syntpsf.f) performs all the above operations in a single run and produces an output file in PGM format of the resulting PSF. The input parameters required are:

1. the field point number (0 to 15);
2. the spectral coefficients $s_1$ to $s_4$;
3. the sample sizes $k_\xi$ and $k_\eta$ (integer number of pixels);
4. the TDI smearing width $\kappa$ (an integer number of pixels);
5. the name of the output file.

The program looks for the relevant files pbXX.YYY in the current directory. Examples of synthesised PSFs are shown in Fig. 11.
Acknowledgement:
thanks to Erik Høg for valuable suggestions and for urging me to explain things better.

9 Availability of files

The 128 PSF files described in Sect. 7 are available in compressed form (total size about 8.5 MB) by anonymous ftp to nastol.astro.lu.se (IP 130.235.11.88). The files are found in the directory incoming/lemnat/SAG-LL-025-dir.

The synthesis program synpts.f is also found in that directory, as well as a simple program (readpgm.f) to read the PSF files and print out a section of it. Figures 10 and 11 were produced with variants of readpgm.f.

Note that the PSF files and programs may be subject to change as specifications and requirements evolve, and will at some time be removed from the ftp area; please refer to the readme.txt file in the same directory for updated information.

Figure 11. Sample data from the synthesised PSF of a K3III star (V = 1.2; s1 = 0.8, s2 = 0.12, s3 = 0.48, s4 = 0.40) observed with the ASM (k1 = k2 = k = 2) at field points 13, 14 and 15 (top to bottom). The PSF is sampled at 1 x 1 pixel resolution, i.e. in steps of half a sample in each direction. These PSFs were calculated with the program synpts.f described in Sect. 8.