Calculation of chromatic displacement

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1 Introduction

This note describes the calculation of chromaticity by the code chrom.f using the full two-dimensional wavefront error (WFE) specified by MMS. Results for a specific WFE map are also given. In contrast with SAG-LI-016 (where a different, one-dimensional WFE was considered) the chromatic displacement shows a simpler, near-monotonic dependence on colour.

2 Calculation of the Line Spread Function

Let \((x, y)\) be coordinates in the pupil plane and \((\xi, \eta)\) angular coordinates in the image plane. For a given wavefront error map \(w(x, y)\), valid at a specific point in the field, the monochromatic optical point spread function (PSF) is given by

\[
P_{\lambda}(\xi, \eta) = \lambda^{-2} \left| \int_{-H/2}^{+H/2} \int_{-D/2}^{+D/2} \exp \left[ \frac{i2\pi}{\lambda} (x\xi + y\eta + w(x, y)) \right] dx dy \right|^2
\]

where \(D = 0.7\) m and \(H = 1.7\) m are the dimensions of the pupil along \(x\) and \(y\). In terms of the complex optical transfer function (OTF), \(O_{\lambda}\), this can be written

\[
P_{\lambda}(\xi, \eta) = \lambda^{-2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} O_{\lambda}(u, v) \exp \left[ \frac{i2\pi}{\lambda} (u\xi + v\eta) \right] du dv
\]

where

\[
O_{\lambda}(u, v) = \int_{-(D-|u|)/2}^{+(D-|u|)/2} \int_{-(H-|v|)/2}^{+(H-|v|)/2} \exp \left[ \frac{i2\pi}{\lambda} \left( w(x + \frac{u}{2}, y + \frac{v}{2}) - w(x - \frac{u}{2}, y - \frac{v}{2}) \right) \right] dx dy
\]

Note that \(O_{\lambda}(u, v) = 0\) for \(|u| \geq D\) or \(|v| \geq H\). In our case we only need the line spread function (LSF) in the scanning direction,

\[
L_{\lambda}(\xi) = \int_{-\infty}^{+\infty} P_{\lambda}(\xi, \eta) d\eta
\]

From (2) we have

\[
L_{\lambda}(\xi) = \lambda^{-1} \int_{-\infty}^{+\infty} O_{\lambda}(u) \exp \left[ \frac{i2\pi}{\lambda} u\xi \right] du
\]

where

\[
O_{\lambda}(u) \equiv O_{\lambda}(u, 0) = \int_{-H/2}^{+H/2} \int_{-(D-|u|)/2}^{+(D-|u|)/2} \exp \left[ \frac{i2\pi}{\lambda} \left( w(x + \frac{u}{2}, y) - w(x - \frac{u}{2}, y) \right) \right] dx dy
\]
Again note that $O_0(u) = 0$ for $|u| \geq D$. Moreover, $O_0(-u) = O_0^*(u)$, where the asterisk signifies the complex conjugate. It is thus sufficient to calculate the (complex) function $O_0(u)$ for $0 \leq u < D$. Then

$$L_0(\xi) = \frac{2}{\lambda} \int_0^D \left\{ \Re[O_0(u)] \cos \left( \frac{2\pi}{\lambda} u \xi \right) - \Im[O_0(u)] \sin \left( \frac{2\pi}{\lambda} u \xi \right) \right\} du \quad (7)$$

The smearing of the LSF by the pixel width ($\Delta \xi$) and by the image motion during the dwell period ($\Delta \xi/m$, where $m$ is the effective number of phases in the charge transfer) is easily taken into account: the OTF $O_0(u)$ is just multiplied by the corresponding window function, $\text{sinc}(\pi u \Delta \xi / \lambda) \text{sinc}(\pi u \Delta \xi / m \lambda)$, prior to the integration in (7).

For a given spectral distribution $f_\lambda$ and instrument characteristics $T_\lambda Q_\lambda$, the polychromatic LSF is calculated as the flux-weighted mean

$$L(\xi) = \frac{\int_0^{\infty} L_\lambda(\xi) Q_\lambda T_\lambda f_\lambda d\lambda}{\int_0^{\infty} Q_\lambda T_\lambda f_\lambda d\lambda} \quad (8)$$

The normalisation of $L_\lambda$ and $L$ is such that the integral over $\xi$ equals the pupil area, $DH$.

3 Location estimator

Having calculated $L(\xi)$ for a given WFE map and spectral distribution, we need to decide where the 'centroide' of the image is. MMS has used the maximum (mode) of the diffraction peak in their theoretical investigation of the chromaticity. In practice, a wider part of the LSF must be used in order to reduce the sensitivity to noise. The actual centroiding algorithm to be used in the GAIA data processing is yet to be defined. For the present investigation the centroid is defined as the maximum of the cross-correlation between the LSF and a gaussian function of given standard width $s$. The centroid $\xi_0$ is then the solution to the equation

$$\int_{-\infty}^{+\infty} L(\xi) (\xi - \xi_0) \exp \left[ -\frac{(\xi - \xi_0)^2}{2s^2} \right] d\xi = 0 \quad (9)$$

For a complicated LSF this may have multiple solutions. The search for a solution should therefore start at the mode of the LSF.

The 'gaussian' location estimator (9) has some nice general properties. Provided the parameter $s$ is selected similar to the standard width of the actual LSF (e.g., $s \approx 0.4 \times \text{FWHM}(L)$), the estimator is robust and not too far from optimal. For small $s$ the location approaches the mode of the LSF, while for large $s$ it becomes the centre of gravity.

4 Practical implementation

The WFE map is defined by the coefficients of the Zernike polynomials, which are taken from an input file. For the discrete wavelengths $\lambda = 300, 302, \ldots, 1050$ nm, the monochromatic OTF is then computed by direct integration of (6). A discretization step of 0.01 m in $x$ and $y$ was found necessary to give the OTF, and hence the chromatic displacement,
with 3–4 significant digits. Since this calculation takes a considerable time (few hours),
the resulting OTF is stored in a binary file for future reference. One such OTF file is
required for each WFE map.

For the same wavelengths, the monochromatic LSFs are then computed by direct integra-
tion of (7). The smearing by pixel width and TDI is included at this step. A selection
of the monochromatic LSF (for \( \lambda = 300, 400, \ldots 1000 \) nm) are saved on a text file for
plotting.

Finally the monochromatic LSFs are folded with a set of spectral distributions according to
(8). The spectra used are the 175 Gunn & Stryker spectra, plus an extinction of \( A_V = 0, 2
\) and 4 mag. For each resulting polychromatic LSF, the mode (peak location) is computed
as well as the gaussian centroid [equation (9)] with \( s = 0.5, 1.0 \) and 2.0 pixel. These
results are put on a text file. For each spectral distribution the index \( H(-6, -4) \) and the
approximate Sloan colour indices \( g' - r', r' - i' \) and \( i' - z' \) are also computed and put on
the output file. (These are not the ‘true’ Sloan colour bands, but rectangular bands with
similar central wavelengths and FWHM.) This allows the relation between the centroid
position and colour indices to be studied off-line.

Direct numerical integration of (6) and (7) rather than using the Fast Fourier Transform
is probably computationally inefficient. However, it may not be so bad if only a small
(central) part of the LSF is needed — in our case, \( \pm 8 \) pixels — rather than the extended
PSF. Moreover, such a simple-minded direct calculation may provide a valuable check of
alternative implementations.

5 Sample results

The sample results given below were obtained with the WFE map for the case ‘Lam50M2’,
field point 15 (MMS note distributed at SAG-10). The Zernike coefficients (Z1 to Z16) for
this map are: \(-0.2142, 0.1599, -0.0032, -0.1864, 0.0101, 0.2521, 0.0685, -0.0686, 0.0342,
0.0698, -0.0245, -0.0030, -0.0336, -0.0068, -0.0146, 0.0064 \). It is a fairly extreme case
(at the corner of the field); the RMS wavefront error is about 100 nm.

Figure 1 shows the monochromatic LSF for \( \lambda = 300, 400, \ldots 1000 \) nm. Pixel and TDI
smearing by \( \Delta \xi = (9 \mu m)/(50 \) m and \( m = 1 \) was applied.

Folding with the standard (CCD#3) quantum efficiency curve and with the three different
stellar spectra gives the sample polychromatic LSF in Fig. 2.

For the complete set of \( 3 \times 175 \) spectral distributions, Fig. 3 gives the peak position versus
the index \( H(-6, -4) \), while Figs. 4–7 give the gaussian centroid (\( s = 1.0 \) pixel) versus
\( g' - r', r' - i', i' - z' \) and \( (g' - r') + 2(r' - i') + (i' - z') \). The latter index was found by trial
and error to provide a more unique relation than the simple indices.

In the previous note, SAG-LI-016, the relation between centroid position and colour was
more complex than found here. It is not clear if this was due to the particular WFE map
assumed in that note, or perhaps to a bug in the old calculations. In any case, the present
results are still not consistent with the MMS relation \( \delta z' / f = K \times H \). For the WFE map
used here, MMS give \( K = +2939.9 \) \( \mu as \), while the slope of the relation in Fig. 3 is about
\( +10000 \) \( \mu as \).
Figure 1. Monochromatic LSF for $\lambda = 300, 400, \ldots, 1000$ nm (the shortest wavelength gives the highest peak).

Figure 2. Polychromatic LSF for three spectral distributions: (A0V, $A_V = 0$), (K3III, $A_V = 0$), and (K3III, $A_V = 4$ mag). The peak is displaced towards the left for the redder stars.
Figure 3. Centroid position (defined as the maximum of the polychromatic LSF) versus the spectral index $H(-6,-4)$. MMS predicted a linear dependence on this index.

Figure 4. Centroid position (defined as the maximum of the cross-correlation with a gaussian of standard width $s = 1.0$ pixel) versus the colour index $g'-r'$. 
Figure 5. Centroid position (defined as the maximum of the cross-correlation with a gaussian of standard width $s = 1.0$ pixel) versus the colour index $i' - i'$.

Figure 6. Centroid position (defined as the maximum of the cross-correlation with a gaussian of standard width $s = 1.0$ pixel) versus the colour index $i' - z'$. 

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Figure 7. Centroid position (defined as the maximum of the cross-correlation with a gaussian of standard width $s = 1.0$ pixel) versus the quantity $(g' - r') + 2(r' - i') + (i' - z')$. Compared with the previous figures this quantity gives a much tighter correlation with the centroid location. This indicates that the whole wavelength range (represented by the colour bands $g'$, $r'$, $i'$ and $z'$) is needed to calibrate the chromaticity.