Propagation of variance estimates through the NDAC processing

L. Lindegren
Lund Observatory

This note summarizes how I understand the estimation of variances are made throughout the main chain of the astrometric reductions: IDT processing, Great-Circle Reduction, and Sphere Solution.

IDT Processing

The basic input data the raw (decompressed) photon counts \( N_k \), which are assumed to follow the Poisson law. The binning of the counts according to their reference phases \( p_k \) results in the total counts \( U_m \) \((m = 1, 2, \ldots, 12)\) which are used for fitting the five signal parameters. This fitting is basically a weighted least-squares solution, in which the weight assigned to the \( m \)-th bin is \( 1/(U_m + 1) \). [Simulations have shown that using \( 1/(U_m + 1) \) instead of \( 1/U_m \) gives a more unbiased estimate of the covariance matrix, cf. NDAC/LO/096, in addition to taking care of the (rare but possible) cases when \( U_m = 0 \).]

The application of these weights is equivalent to multiplying each observation equation with \((1 + U_m)^{-1/2}\), resulting in a set of equations in which the expected noise term has unit variance. The inverse of the corresponding \( 5 \times 5 \) normal equations matrix gives an a priori covariance matrix which we denote \( C_0 \). The sum of the squares of the residuals, SSR, is expected to follow the \( \chi^2 \) distribution with 12 - 5 = 7 degrees of freedom. One of the statistics calculated is the unit weight variance, UWV = SSR/7, which should be close to one. According to van Leeuwen the UWV is in fact nearly always slightly greater than unity, indicating the presence of some unmodelled effects (errors in \( p_k \), third harmonic, background radiation, compression/decompression, ...?). Consequently it can be inferred that the covariances in \( C_0 \) are normally underestimated. For this reason a corrected covariance matrix for the five signal parameters is computed as

\[
C = C_0 \times \text{UWV}.
\]

(1)

In particular the standard errors of the grid coordinates \( \sigma_g \) transferred to Copenhagen are directly derived from the corrected matrices \( C \).

Great-Circle Reductions

The grid coordinates, with their associated standard errors, are combined by the weighted least-squares method in the (geometrical) great-circle solution. This gives the geometrical attitude and the geometrical abscissae, with their associated standard errors from the inverse normal matrix. The geometrical abscissae are however not further used. Instead, a smoothed solution is obtained by a special post-processing: a spline function is fitted to the geometrical attitude, and the resulting smoothed attitude is then again combined with the grid coordinates to give the 'smoothed' abscissae. Unfortunately with this method it has not been possible to compute rigorously the variances of the resulting abscissae. A variance estimate is nevertheless produced, basically by using in the post-processing the following estimate of the variance of the grid coordinate relative to the spline:
\[ \sigma_{\text{g, sm}}^2 = \sigma_g^2 + 4\sigma_0^2. \]  

(2)

Here \( \sigma_0 \) is the formal error of the spline fit to the spin phase updates of the relevant frame. The factor 4 accounts for the fact that the attitude error contained in the spline fit is correlated over several frames (while the grid coordinate errors are essentially uncorrelated); this number was determined empirically and no theoretical significance should be attached to it. The actual ascissa variance estimate can then be written (with some simplifying approximations):

\[ \sigma_{\text{absc, sm}}^2 = \left[ \sum_{\text{obs}} \sigma_{g, \text{sm}}^{-2} \right]^{-1} \times \text{UWV} \]  

(3)

where the UWV is calculated from the residuals of the spline fit, normalized by \( \sigma_g, \text{sm} \), and the corresponding degrees of freedom (number of observations minus number of stars minus number of spline coefficients). The output statistic called the ‘normalized mean error’ is in fact the square root of the UWV. Typically the NME falls in the range 1.00 to 1.05, indicating a fairly satisfactory level of modelization errors.

It is known that Eq. (3) badly underestimates the ascissa variances especially for bright stars. Actually a further constant term on the right side of Eq. (2), of the order of 100–200 mas\(^2\), would probably give more realistic estimates of \( \sigma_{\text{absc, sm}} \), but that would on the other hand cause problems both with the weight reduction scheme (to reject outliers) and the calculation of the UWV. For that reason it has been decided to keep the formulation in Eqs. (2)–(3) and instead make a posterior correction of the variances in the sphere solution.

Sphere Solution

The sphere solution is a rigorous least-squares solution in which the individual observations (in this case, the ascissae) are weighted according to their assumed variances. These variances are computed as:

\[ \sigma_a^2 = \sigma_{\text{absc, sm}}^2 + \sigma_0^2 \]  

(4)

with (currently) \( \sigma_0 = 2.70 \text{ mas} \). The latter value was chosen in order to give a reasonable overall distribution of residuals in the one-year sphere solution. More specifically, the procedure was as follows. For each star in the sphere solution, a \( \chi^2 \) statistic is computed from the (normalized) residuals. Depending on the degrees of freedom for each star (equal to the number of ascissa observations of the star minus the number of astrometric parameters), the \( \chi^2 \) variable is transformed to a statistic \( t_1 \) which would be \( N(0,1) \) (unit normal) for the theoretical \( \chi^2 \) distribution. The resulting distribution of \( t_1 \) will of course depend on the chosen \( \sigma_0 \), since this affects the weight given to each ascissa. The global distribution of \( t_1 \) is very skew, with a large excess of high (positive) values due to double stars and outlier points. The adopted \( \sigma_0 = 2.70 \text{ mas} \) is such that the mode of the distribution (being relatively insensitive to the long tail of positive values) occurred at \( t_1 = 0 \).

A detailed examination of the external errors as function of magnitude and the formal errors is in progress, and may lead to a refinement of \( \sigma_0 \) to be used in subsequent sphere solutions. Possibly this refined \( \sigma_0 \) will be a function of magnitude.

Finally it should be recalled that, once \( \sigma_{\text{absc, sm}} \) has been fixed for each observation, the final covariance matrix for the astrometric parameters is completely determined by the geometry and temporal distribution of the observations. As a consequence, the uncertainty in the relations between the various elements of the covariance matrix is quite small. This applies in particular to the ratios between the mean errors of the different parameters of the same star, and to the correlation coefficients between them.