RELATION BETWEEN FAST AND NDAC ATTITUDE REPRESENTATIONS

by L Lindegren

1. Introduction

For the exchange of simulated input to the Great Circle Reduction and for comparing results of this process (and eventually also the attitude reconstruction process) it will be necessary to transform between the different attitude representations used by FAST and NDAC. This note summarizes the definitions used by the two consortia and derives the required transformations. For the FAST attitude, I consider only the representation by Eulerian 3-2-1 angles relative to the RGC system, i.e.

\[ \psi = (\text{PSI})_e \]
\[ \Theta = (\text{TETA})_e \]
\[ \varphi = (\text{PHI})_e \]

(FAST Interface Document, File ACFRA, p. 9).

2. Celestial reference systems

FAST uses the ecliptical system, equinox J2000, while NDAC uses the equatorial (same equinox). The consortia agree on using the IAU (1976) value for the obliquity of the ecliptic at J2000, viz.

\[ \epsilon = 23^\circ 26' 21.448'' \]

(rather than, for instance, the value used for preparation of the JPL ephemerides). Note that \( \epsilon \) is an exact (conventional) constant. The transformation between the two systems is also exact and given by

\[ K = \frac{N A}{1}(\epsilon) \]

(3)

where \( K = [l \; i \; k] \) is the ecliptical triad (with unit vector \( l \) towards the vernal equinox, \( k \) towards the north ecliptical pole, and \( i = k \times l \).
\( N = [n, m, p] \) is the equatorial triad with \( n \) towards the north equatorial pole. Multiplication by the 3x3-matrix \( A_1(\varepsilon) \) effects a rotation about the first axis by the angle \( \varepsilon \). *)

In addition, FAST uses a local celestial system \( \Sigma_{RGC} \) defined by a fixed Reference Great Circle (RGC) and its intersection with the ecliptic. I shall denote this system by the triad \( R = [p, q, r] \), where \( r \) is the (positive) pole of the RGC and \( p \) the ascending node on the ecliptic. \( R \) is completely specified by the ecliptical longitude and latitude of \( r \): \( \lambda_r, \beta_r \). It is readily seen that \( R \) is obtained from \( N \) by two rotations:

(i) by the angle \( \lambda_r + \frac{1}{2} \pi \) about the third axis;
(ii) by the angle \( \frac{1}{2} \pi - \beta_r \) about the first axis.

Thus,

\[ R = KA_2(\lambda_r + \frac{1}{2} \pi)A_1(\frac{1}{2} \pi - \beta_r) \]  \hspace{1cm} (4)

NDAC on the other hand uses an intermediary 'heliotropic' system \( S = [s, t, k] \), in which the first axis \( s \) is directed towards the 'nominal sun', \( k \) is the ecliptical pole, and \( t = k \times s \). The nominal sun is exactly on the ecliptic J2000, and its ecliptical longitude \( \lambda_s \) is given as function of time by an exact (conventional) formula:

\[ \lambda_s = L + 2e \sin g + 1.25 e^2 \sin 2g \]  \hspace{1cm} (5a)
\[ L = -1.38691 + 0.0172021240 \, \text{d} \]  \hspace{1cm} (5b)
\[ g = -0.04114 + 0.0172019696 \, \text{d} \]  \hspace{1cm} (5c)
\[ e = 0.016714 \]  \hspace{1cm} (5d)

*) The matrices for rotation about the first, second, and third axis are:

\[ A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}, \quad A_2 = \begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix}, \quad A_3 = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \]  \hspace{1cm} (*)

where \( c = \cos(\text{angle}) \), \( s = \sin(\text{angle}) \). Note that this definition is opposite to the usual rotation matrices (e.g., Eichhorn, *Astronomy of Star Positions*, p. 11), which transform coordinates rather than vectors.
where \( d \) is the time in SI seconds from 1988 Jan 1, 12\(^{h}\)00\(^{m}\)00\(^{s}\) UTC, divided by 86400 (cf. IF-3-00-4, Issue 5, rev. 3, p. 23, and discussion at the 14th HST Meeting, September 1986). The definition (5) has been chosen such that the satellitocentric proper direction to the real sun is not more than about 1 arcmin from the nominal direction \( \mathbf{s} \). Clearly \( \mathbf{s} \) is obtained from \( \mathbf{k} \) by a single rotation by the angle \( \lambda_s \) about the third axis; thus:

\[
\mathbf{s} = K A_3(\lambda_s)
\]  

(6)

It must be emphasized that \( \mathbf{r} \) and \( \mathbf{s} \) are obtained by exact transformation of \( \mathbf{k} \) [or \( \mathbf{n} \), via (3)], given the adopted values for \( (\lambda_r, \beta_r) \) and \( \lambda_s \), respectively.

3. Instrument system

Both FAST and NDAC define a reference system directly linked to (observable properties of) the optics and grid. The attitude describes the orientation of this instrument system with respect to any of the celestial reference systems mentioned above.

In FAST terminology, the 'intermediary moving celestial frame' \( E_M \) is defined by the great circle through the two projections on the celestial sphere of the active starmapper apex, and by an origin \( (\mathbf{x}) \) on that circle. The origin bisects the acute angle between the two points on the circle where it is intersected by the projections of the central slit of the main grid. Let us define the instrument triad \( \mathbf{z} = [\mathbf{x} \, \mathbf{y} \, \mathbf{z}] \) with \( \mathbf{x} \) towards the origin \( \mathbf{x} \), \( \mathbf{z} \) normal to the great circle through the starmapper apex (with \( +z \) approximately 43° from the sun), and \( \mathbf{y} = \mathbf{z} \times \mathbf{x} \). Provided that the consortia agree on the definition of the 'central slit' (unfortunately there is an even number of slits on the main grid), our systems will be the same at least down to the level of 0.01 arcsec.

4. Definition of attitude angles

FAST relates the instrument triad \( \mathbf{z} \) to the RGC system \( \mathbf{r} \) by means of the 3-2-1 Euler angles \( \Psi, \Theta, \Phi \): 

\[
\mathbf{z} = R A_3(\Psi) A_2(\Theta) A_1(\Phi)
\]  

(7)
NDAC relates \( Z \) to the heliotropic system \( \mathbb{S} \) by means of the 1-2-3 Euler angles \( \nu - \frac{1}{2} \pi, \frac{1}{2} \pi - \xi, \Omega \):

\[
Z = S A_1(\nu - \frac{1}{2} \pi) A_2(\frac{1}{2} \pi - \xi) A_3(\Omega) \tag{8}
\]

The 3-2-1 and 1-2-3 Euler angle rotations have singularities when the second angle is \( \pm \frac{1}{2} \pi \), and a corresponding ambiguity of the second angle. This is resolved in the present cases by requiring that \( |\theta| < \frac{1}{2} \pi \) and \( \frac{1}{2} \pi - \xi < \frac{1}{2} \pi \) (or \( 0 < \xi < \pi \)).

5. Relation between attitude angles

Equating (7) and (8) and premultiplying by \( R' \) we have

\[
A_3(\psi)A_2(\Theta)A_1(\phi) = R'S A_1(\nu - \frac{1}{2} \pi) A_2(\frac{1}{2} \pi - \xi) A_3(\Omega) \tag{9}
\]

or the matrix equation

\[
B(\psi, \Theta, \phi) = C(\lambda_r, \beta_r, \lambda_s)D(\nu, \xi, \Omega) \tag{10}
\]

with

\[
B(\psi, \Theta, \phi) = A_3(\psi)A_2(\Theta)A_1(\phi) = \begin{bmatrix}
\cos \Theta & -\psi \sin \Theta & \psi \cos \Theta + \psi \sin \Theta \\
\sin \Theta & \psi \cos \Theta + \psi \sin \Theta & -\psi \sin \Theta + \psi \cos \Theta \\
-\sin \phi & \cos \phi & \cos \phi
\end{bmatrix} \tag{11}
\]

\[
C(\lambda_r, \beta_r, \lambda_s) = R'S = A_1(\beta_r - \frac{1}{2} \pi) A_3(-\lambda_r - \frac{1}{2} \pi) A_2(\lambda_s) = A_1(\beta_r - \frac{1}{2} \pi) A_3(\lambda_s - \lambda_r - \frac{1}{2} \pi)
\]

\[
= \begin{bmatrix}
s \Delta & c \Delta & 0 \\
-s \beta_r c \Delta & s \beta_r c \Delta & c \beta_r \\
c \beta_r c \Delta & -c \beta_r c \Delta & s \beta_r
\end{bmatrix} \quad (\Delta = \lambda_s - \lambda_r) \tag{12}
\]

\[
D(\nu, \xi, \Omega) = A_1(\nu - \frac{1}{2} \pi) A_2(\frac{1}{2} \pi - \xi) A_3(\Omega) = \begin{bmatrix}
\sin \xi \Omega & -\sin \xi \Omega & \cos \xi \\
-\cos \xi \Omega + \sin \xi \Omega & \cos \xi \Omega + \sin \xi \Omega & \cos \xi \Omega \\
-\sin \xi \Omega - \cos \xi \Omega & \sin \xi \Omega - \cos \xi \Omega & \sin \xi \Omega
\end{bmatrix} \tag{13}
\]
To convert from NDAC angles \((v, \xi, \Omega)\) to FAST \((\psi, \Theta, \Omega)\), the following procedure can be used:

(i) compute the \(3\times3\) matrix \(C\) from given \(\lambda_x, \beta_x, \lambda_s\) [eq (12)];

(ii) compute the \(3\times3\) matrix \(D\) from \(v, \xi, \Omega\) [eq (13)];

(iii) compute \(B = CD\) (or at least the elements of \(B\) used below);

(iv) then

\[
\begin{align*}
\psi &= \text{atan2}(B_{21}, B_{11}) \\
\Theta &= \arcsin(-B_{31}) \\
\varphi &= \text{atan2}(B_{32}, B_{33})
\end{align*}
\]

\((14)\)

Note that the arcsine can be used for \(\Theta\) without ambiguity, since \(|\Theta| < \frac{\pi}{2}\).

To convert from FAST \((\psi, \Theta, \varphi)\) to NDAC \((v, \xi, \Omega)\), the procedure is:

(i) compute \(C\) from \(\lambda_x, \beta_x, \lambda_s\) [eq (12)];

(ii) compute the \(3\times3\) matrix \(D\) from \(\psi, \Theta, \varphi\) [eq (11)];

(iii) compute \(D = C'B\) (or at least the elements of \(D\) used below), where \(C'\) is the transpose of \(C\);

(iv) then

\[
\begin{align*}
v &= \text{atan2}(D_{33}, D_{23}) \\
\xi &= \arccos(D_{13}) \\
\Omega &= \text{atan2}(-D_{12}, D_{11})
\end{align*}
\]

\((15)\)

Note that the arccos can be used without ambiguity, since \(0 < \xi < \pi\).

6. The real-time attitude angles

It may be useful to review here the definition of the RTAD angles in the framework of the DRC definitions. The RTAD angles \(\varphi_C, \Theta_C, \psi_C\) are Eulerian 1-2-3 angles with respect to the nominal spacecraft reference system (given by the nominal scanning law). Thus we have still another set of Euler
angles related to one more celestial reference system. A further complication is that the 'attitude control reference frame' \( Z_G = [X_G Y_G Z_G] \) is not exactly the same as the DRC instrument system \( Z \). The differences are:

(i) \( Z_G \) is normal to the viewing plane through the projections of main grid centre \( O_G \), whereas \( Z \) is normal to the plane through the projections of the active starmapper apex;

(ii) \( X_G \) is the point midway between the two projections of \( O_G \), whereas \( X \) is midway between the intersections of the central slit with the great circle through the starmapper apex.

\( Z \) and \( Z_G \) would coincide if (i) the grid had its nominal orientation with respect to the beam combiner and (ii) the central slit went through the grid centre \( O_G \). Unfortunately, the grid orientation may be wrong by up to a few arcmin, and the central slit (whichever is chosen) will not go through the centre of the grid. Thus we must consider the transformation from \( Z \) to \( Z_G \) in terms of observable quantities.

In the grid reference frame \((G, H)\) (converted to radians through division by the focal length) the coordinates of the grid centre are \( G = H = 0 \), and of the starmapper apex

\[
G_a \approx 0.01543 \, q, \quad H_a = 0 \quad (16)
\]

where \( q \) is the sign of the \( G \) or \( H \) coordinate of the active starmapper \((q = 1 \) for the 'following' starmapper, i.e. if the image reaches the starmapper after the main grid for the nominal sense of spin). Also, let \( G_c \) be the coordinate of the central slit. Nominally, \( G_c = \pm 0.604 \) arcsec depending on which slit is chosen as 'central'. The orientation error of the grid may be given by the angle \( \mu \) from the ideal \((G,H)\) to the actual \((G,H)\) \((SS.6.01.0, Issue 5, pp 31 and 34)\). In terms of the NDAC field-to-grid transformation coefficients, \( \mu \approx \varepsilon_0/\varepsilon_1 \) to sufficient accuracy. The situation is schematically depicted in Figure 1. It is seen that \( Z \) is (to first order in the small angles involved) obtained from \( Z_G \) by

\[
Z = Z_G A_3(-G_c) A_2(qe), \quad e \approx 0.01543 \, \mu \sec(\gamma) \quad (17)
\]

where \( e \) is practically a constant.
The celestial reference system used for the attitude determination is given by the nominal orientation of $Z_G$ at a certain instant, i.e. according to the nominal scanning law. We denote this system $Z_N$. It is obtained exactly as (8) but with nominal heliotropic angles instead of actual:

$$Z_N = S_{A_1}(\nu_N - i\pi)A_2(4\pi - \varepsilon_N)A_3(\Omega_N)$$

where $\nu_N$, $\varepsilon_N$, $\Omega_N$ are known functions of time according to the nominal scanning law. The RTAD angles then give the actual orientation of $Z_G$ as

$$Z_G = Z_NA_1(\psi_G)A_2(\Theta_G - qe)A_3(\psi_G - G_C)$$

Since $G_C$ and $e$ are small angles, we find to sufficient accuracy, by a combination of (17) and (19),

$$Z = Z_NA_1(\psi_G)A_2(\Theta_G - qe)A_3(\psi_G - G_C)$$

Thus we should simply use the modified RTAD angles $\Theta_G^* = \Theta_G + qe$, $\psi_G^* = \psi_G - G_C$ instead of $\Theta_G$, $\psi_G$ in order to get to the DRC instrument system.

Conversion of $\psi_G$, $\Theta_G$, $\psi_G$ into the actual heliotropic angles $\nu$, $\varepsilon$, $\Omega$ is straightforward after equating (8) with (18)+(20). A convenient algorithm was derived in NDAC/LO/043, eqs (14)-(15).

7. Numerical example

For the time $d = 1000.0$ days, $\lambda_s = 3.2157291662$, $\nu_N = 0.5245575761$, $\varepsilon_N = 0.7504915784$, $\Omega_N = 2.4639297917$.

Assuming $\psi_G = 0.0019$, $\Theta_G^* = -0.0011$, $\psi_G^* = 0.0026$ (exactly), I obtain the following NDAC attitude angles:

$\nu = 0.5233959239$

$\varepsilon = 0.7484437196$

$\Omega = 2.467379137$

and, relative to the RGC pole $\lambda_r = 3.90$, $\beta_r = 0.35$ (exactly), the FAST angles

$\psi = -0.2765211609$

$\Theta = -0.0054158705$

$\phi = 0.0045901399$
FIGURE 1. Relation between the attitude control reference frame $Z_G = [x_G, y_G, z_G]$ and the DRC instrument system $Z = [x, y, z]$. (View from 'outside' the celestial sphere.) $O_G$ = centre of main grid, $C$ = central slit, $A$ = apex of active star-mapper ($q = 1$ shown), $VP$ = viewing plane, $ISC$ = instantaneous scanning circle, $\mu_G$ = grid orientation error.