Light-time effects and the modelling of stellar proper motion

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The variation of the barycentric coordinate direction to a star is modelled under the assumption that the star has constant space velocity. The usual formulae for computing the direction as function of time (e.g. The Astronomical Almanac for 1984, page B39) do not include light-time effects, or at least no explicit distinction between the epochs of light emission at the star ($T_*$) and reception at the barycentre ($T$). Murray (VA, Section 2.6.4) does make this distinction, but derives the Taylor expansion of the direction (to second order in $T$) under the assumption $|d^2b/dT^2| = 0$, where $b$ is the coordinate of the star, which is not quite equivalent to constant space velocity, $|d^2b/dT_*^2| = 0$. The following is an attempt to include light time rigorously in a flat metric (ignoring the relativistic delay).

Let $b(T_*)$ be the barycentric coordinate of the star. Light emitted from the star at epoch $T_*$ will reach the Solar System barycentre at $T = T_* + c^{-1}|b(T_*)|$. The astrometric parameters at the latter epoch are defined in terms of the barycentric direction, proper motion, distance, and radial velocity

$$m = \langle b(T_*) \rangle \quad (1a)$$

$$\mu = \frac{d}{dT} \langle b(T_*) \rangle \quad (1b)$$

$$r = |b(T_*)| \quad (1c)$$

$$\rho = \frac{d}{dT} |b(T_*)| \quad (1d)$$

These are all functions of $T$ with

$$T_* = T - r(T)/c \quad (2)$$

The space velocity is
\[ y = \frac{d}{dT_\star} b(T_\star) \]

\[ = \frac{d}{dT} \left[ \frac{\dot{m}(T)r(T)}{m(T)} \right] \frac{dT}{dT_\star} \]

\[ = (\mu r + m_0)(1 - \rho_0/c)^{-1} \tag{3} \]

According to our assumption, this vector is independent of T and can in particular be calculated from the astrometric parameters at the reference epoch \( T_0 \), viz. \( m_0 = m(T_0) \) etc. With \( T_{\star 0} = T_0 - r_0/c \) we then have for the arbitrary epoch \( T = T_0 + \tau \):

\[ m(T_0 + \tau) = < b(T_{\star 0}) + y[T_\star - T_{\star 0}] > \]

\[ = < \frac{\dot{m}r_0}{m_0} + (\mu_0 r_0 + m_0 \rho_0)(1 - \rho_0/c)^{-1}(T_\star - T_{\star 0}) > \]

\[ = < \frac{\dot{m}}{m_0} + (\mu_0 + m_0 \rho_0/r_0)(1 - \rho_0/c)^{-1}\tau_\star > \tag{4} \]

where

\[ \tau_\star = T_\star - T_{\star 0} \]

\[ = \tau - [r(T_0 + \tau) - r(T_0)]/c \tag{5} \]

is the corresponding interval between emissions, obtained by iterating

\[ r(T_0 + \tau) = r_0 \left[ m_0 + (\mu_0 + m_0 \rho_0/r_0)(1 - \rho_0/c)^{-1}\tau_\star \right] \tag{6} \]

If the secular variation of radial velocity is neglected, we have \( r - r_0 = \rho \tau \) in (5). In this approximation, equation (4) becomes the 'classical'

\[ m(T_0 + \tau) = < \frac{\dot{m}}{m_0} + (\mu_0 + m_0 \rho_0/r_0)\tau > \tag{7} \]

and by expansion

\[ m(T_0 + \tau) = m_0 + \mu_0 \tau - [m_0 \mu_0^{-1} + \mu_0 \rho_0/r_0] \tau^2 + O(\tau^3) \tag{8} \]
Equation (8) is identical to (2.6.27) in VA.

In order to derive the complete second-order expression we must however include the variation of radial velocity. Taking the derivative of (3) we have after some simplification

$$0 = \frac{d\mu}{dT} r + \mu [2\rho + (r/c)(1 - \rho/c)^{-1} \frac{d\rho}{dT}] + \mu [1 - \rho/c]^{-1} \frac{d\rho}{dT}$$

(9)

Multiplying scalarly by $m$ and using that $m'\mu = 0$, $m'(d\mu/dT) = -\mu'\mu$, we find

$$\frac{d\rho}{dT} = \mu'\mu r (1 - \rho/c)$$

(10)

Inserting $r - r_o = \rho \tau + \frac{1}{2} (d\rho/dT) \tau^2 + O(\tau^3)$ in (5) we have then

$$(1 - \rho_o/c)^{-1} \tau = \tau - \frac{1}{2} \mu_o'\mu_o (r_o/c) \tau^2 + O(\tau^3)$$

(11)

and consequently

$$m(T_o + \tau) = m_o + \mu_o \tau - [m_o + \mu_o'\mu_o + \mu_o (\rho_o/r_o + \frac{1}{2} \mu_o'\mu_o r_o/c)] \tau^2 + O(\tau^3)$$

(12)

The rate of change of the modulus of the proper motion $\mu = |\mu|$ is

$$\frac{d\mu}{dT} = <\mu^r> \frac{d\mu}{dT}$$

$$= -2\mu \rho/r - \mu^3 r/c$$

(13)

[cf. VA, (2.6.29)]. The second term is typically a factor $\sim v/c$ smaller than the first (if $\mu r \sim \rho$); e.g. for Barnard's star, $2\mu \rho/r = 1.24 \text{ mas}/\text{yr}$ and $\mu^3 r/c = 0.15 \text{ mas}/\text{yr}^2$. It can be noted that the 'apparent' acceleration

$$\frac{d^2\mu}{dT^2} = - (\mu r + \mu \rho) \mu^2 r/c$$

(14)

of Barnard's star is $6.7 \times 10^{-11} \text{ m s}^{-2}$, that is less than the galactic acceleration ($\sim 2 \times 10^{-10} \text{ m s}^{-2}$). It is therefore safe to neglect light-time effects altogether in modelling stellar proper motion, and thus to use the simple formula (7).