Small-scale experiments in Step 2/3 with elimination of the astrometric parameters. I. Solutions without global parameters.

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1. Introduction

In a previous note (NDAC/LO/024), we reported some experiments with the "old" PRS-method, eliminating the set zero-points and solving for the PRS astrometric parameters. The present note is a preliminary discussion of new experiments with pre-adjustment and elimination of the astrometric parameters instead, as envisaged in the WP 6100 and 7100 definitions (NDAC/LO/021-22).

The HP 9000 computer of Lund Observatory allows primary memory reduction of (double precision) normal equations smaller than some 240x240. We have chosen therefore to simulate one year of observations (no proper motions) with typically 10000 stars and a realistic FOW (0.9), but including only 100-230 of the 730 (12h) sets. Using the virtual memory capacity, it was possible to process runs with 304 and 365 sets, but with very low efficiency and hence long computing times (5-8 hours).

2. Observation simulation

The real observations proceed set-wise, that is, during successive scans of the satellite a number of stars are observed that together constitute a set. The set-solution process gives resulting abscissae for each set at a time. These abscissae-results are then sorted star-wise before the Step 2/3-process. In the present study, we start directly with an "ordered" (random) star-catalogue. The scanning motion is then simulated, and for each set where the star is observed one observation equation is formed. More specifically, the procedure is as follows.

The "star-catalogue" is never set up as a whole, but instead one star (index 1) at a time is created with a random position and zero parallax. The scanning motion is taken from Lindegren's (1983-10-21) note, with $d_1 = 100$ (observations starting 1988.27). For the chosen selection of sets (normally
of 5 scans = \(10^{14} 10^m\) duration), the scan-poles at beginning, middle and end are stored. The angles \(\psi_1^{(n)}\) and \(\psi_2^{(n)}\) from the star to the beginning and end scan poles of set \(n\) are then computed, and if the conditions

\[
[\max(\cos \psi_1^{(n)}, \cos \psi_2^{(n)}) > -\sin \frac{w}{2}] \quad \text{and} \quad [\min(\cos \psi_1^{(n)}, \cos \psi_2^{(n)}) < \sin \frac{w}{2}]
\]

are both fulfilled (with \(w\) the width of the FOW), the star was observed in set \(n\) (Lindgren 1981-02-06).

For each of these observed sets \((1)\), one observation equation is formed with the differential coefficients \(A_{ij}^{(1)}\), \(A_{ij}^{(2)}\) and \(A_{ij}^{(3)}\) from NDAC/LO/021. The barycentric position of the observer, \(\rho_o\), was taken to be

\[
\rho_o = -s + s_b
\]

with \(s\) the position of the sun from Lindgren (1983-10-21) and \(s_b\) the "typical" (ecliptic) barycentric position of the sun \((0.0035, 0.0008, 0)\). Only one set orientation parameter (the correction to the zero-point) is included, and therefore \(B_{ij}^{(1)} \equiv \frac{1}{n}\). No global parameters are included in the present runs, but they are easily incorporated later. The right-hand sides of the observation equations are given random gaussian abscissa errors with unit standard deviation. The systematic set zero-points were either set to zero or were normally distributed with standard deviation = 500 (Cf. Section 3).

Stars observed in less than four sets can not contribute to the set zero-point determination (because the 3 astrometric parameters need 3 observations), but those with three observations were still included in the ADJAST-tests reported below. This scan-number limit excludes an increasing number of stars (especially at low ecliptic latitudes) when the set-numbers are decreased, see Section 3 below).

3. Accumulation of the normal equations

The normal equations were formed in principle according to Table 1 in NDAC/LO/022, but for practical reasons there are some differences. No slit-errors were simulated, and thus not the MESH-procedure. Also, the "real" ADJAST-procedure with successive updating of the astrometric positions was not used. Instead, the final iteration of ADJAST was simulated by using a pure gaussian rhs (set zero-points \(\equiv 0\)) in the observation equations. It was thought more interesting, however, to simulate the set
zero-point solution with non-zero à priori values. Therefore, a second set of rhs-values ("systematic" zero-points + random noise) was used for forming the full normal equations. The "false" ADJAST also necessitated re-installment of the (now) non-zero ΣA_ij 'h_ij'-term as in the former Table 1 of NDAC/LO/021.

For the ADJAST-study, the number of sets is not limited by the computer capacity, and simulations were done with up to 730 sets. Table 1 shows some typical astrometric mean errors for stars at different ecliptic latitudes. With 365 or 730 sets, these results are similar to those reported in NDAC/LO/017. With only 182 sets included, the astrometric parameters are poorly determined due to the low number of observations per star (mean 3.8 for 0-20° latitude). The general trends of accuracy versus latitude are still recognizable, however.

Table 1. Theoretical mean astrometric errors from ADJAST for stars at different ecliptic latitudes. (Abscissa mean error = 1).

<table>
<thead>
<tr>
<th>ecl. lat.</th>
<th>NSET=182</th>
<th>NSET=365</th>
<th>NSET=730</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>α δ π</td>
<td>α δ π</td>
<td>α δ π</td>
</tr>
<tr>
<td>0 - 20°</td>
<td>10 4 13</td>
<td>2.3 1.5 3.0</td>
<td>0.59 0.43 0.67</td>
</tr>
<tr>
<td>20 - 40</td>
<td>3 3 5</td>
<td>1.0 1.2 1.8</td>
<td>0.50 0.41 0.61</td>
</tr>
<tr>
<td>40 - 60</td>
<td>1.7 2 3</td>
<td>0.51 0.53 0.75</td>
<td>0.34 0.36 0.49</td>
</tr>
<tr>
<td>60 - 90</td>
<td>3 4 5</td>
<td>0.53 0.52 0.59</td>
<td>0.35 0.35 0.39</td>
</tr>
</tbody>
</table>

4. Rank-deficiency of the normal equations

The eigenvalues of the NSET x NSET normal equations were found by programs given by Lawson and Hanson ("Solving Least Squares Problems", Prentice-Hall, 1982). As expected, the three smallest values are at least a factor of 1000 smaller than the others. As a useful single parameter to describe this "jump", we have chosen to define

$$\Lambda = \frac{\lambda_m}{\mathcal{L}}$$

(3)

where $\mathcal{L}$ is the geometric mean of the three smallest eigenvalues and $\lambda_m$ the median of the "large" ones. Table 2 gives typical values for runs with different NSET.
Table 2. Eigenvalues for normal equations from different numbers of sets.

<table>
<thead>
<tr>
<th>NSET</th>
<th>$\lambda_1 \times 1000$</th>
<th>$\lambda_2 \times 1000$</th>
<th>$\lambda_3 \times 1000$</th>
<th>$\lambda_4 \times 1000$</th>
<th>$\lambda_m$</th>
<th>$\lambda_{max}$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>0.85</td>
<td>1.32</td>
<td>6.02</td>
<td>1.89</td>
<td>4.57</td>
<td>17.1</td>
<td>48.9</td>
</tr>
<tr>
<td>122</td>
<td>1.48</td>
<td>1.88</td>
<td>8.29</td>
<td>2.85</td>
<td>6.37</td>
<td>24.6</td>
<td>54.8</td>
</tr>
<tr>
<td>152</td>
<td>2.55</td>
<td>3.08</td>
<td>13.2</td>
<td>4.70</td>
<td>11.6</td>
<td>38.9</td>
<td>127.7</td>
</tr>
<tr>
<td>182</td>
<td>3.36</td>
<td>4.45</td>
<td>17.1</td>
<td>6.35</td>
<td>16.2</td>
<td>51.7</td>
<td>102.9</td>
</tr>
<tr>
<td>230</td>
<td>4.83</td>
<td>6.30</td>
<td>21.5</td>
<td>8.68</td>
<td>24.0</td>
<td>70.2</td>
<td>124.8</td>
</tr>
</tbody>
</table>

As explained in NDAC/LO/024, non-zero $\Delta$-values are due to the finite width of the FOW. Also, the length of the sets plays a role, the scan-pole moving about $0.40$ per scan. The values in Table 2 are for $w = 0.9$ and 5 scans per set. This corresponds to maximum ordinates from the RGC of some $w/2+0.8$. From runs with other $w$-values (NSET=182, 5 scans per set), we found $\Delta$-values that closely (within 5%) followed the related expression

$$\Delta \approx 50000 (w + 1.6)^{-2}$$

(4)

Experiments with only 1 scan per set (NSET=230) gave instead

$$\Delta \approx 35000 (w + 0.1)^{-2}$$

(5)

where the (low) $0.1$-value may be due to the special geometry of a single scan. Qualitatively, the "deficient rank-deficiency" can be well understood, and as will be shown below, it has little effect on the solution.

5. Pseudo-solution of the normal equations

As for a system with full rank-deficiency, the pseudo-solution was obtained by the method given in NDAC/LO/021 and NDAC/LO/018. The zero-points for three sets (not necessarily with orthogonal RGC-poles) are "fixed" to zero, and a standard least-squares solution is made. (A Cholesky program written by Lindegren is used for complete as well as partial reductions.) This solution is then modified using the known "null-space" matrix $\mathcal{W}$. The expression for $\mathcal{W}$ in NDAC/LO/021 is partly in error, however, and the correct "null-vectors" are simply
\[ W = \begin{pmatrix}
-x_{r_1} & -y_{r_1} & -z_{r_1} \\
-\cdots & \cdots & \cdots \\
-x_{r_j} & -y_{r_j} & -z_{r_j} \\
-\cdots & \cdots & \cdots \\
-x_{r_{\text{NSET}}} & -y_{r_{\text{NSET}}} & -z_{r_{\text{NSET}}}
\end{pmatrix} \quad (6) \]

where \((x_{r_j}, y_{r_j}, z_{r_j})\) is the RGC-pole vector for set \(j\). Because the small eigenvalues are not exactly zero, the \(w_k\) are not exact null-vectors. Typically, we have \(|A w_k| \approx 0.1 |w_k|\).

The pseudo-solution thus obtained does very closely reproduce the original set zero-points after these have been rotated to minimum variance. We may form a \(\chi^2\)-measure by summing the squared deviations (solution minus rotated original set zero-point) divided by the pseudo-variances. The observed values in Table 3 are close to the expected NSET/\(2 \cdot \text{NSET}\), showing that these variances give realistic estimates of the set zero-point errors.

The pseudo-variances were computed by Lindegren's (NDAC/LO/018) method, and for comparison by the eigenvalue-programs of Lawson and Hanson. Very satisfyingly, the two methods show no systematic difference, and the random deviations are less than about 0.1 %. From the full pseudo-inverse of the normal equations, it appears also that the correlations between the different set zero-points are less than a few percent.

Table 3. \(\chi^2\)-measure for solutions with different NSET.

<table>
<thead>
<tr>
<th>NSET</th>
<th>46</th>
<th>61</th>
<th>81</th>
<th>101</th>
<th>122</th>
<th>152</th>
<th>182a</th>
<th>182b</th>
<th>230</th>
<th>304</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\chi^2)</td>
<td>39</td>
<td>51</td>
<td>74</td>
<td>77</td>
<td>119</td>
<td>136</td>
<td>154</td>
<td>195</td>
<td>246</td>
<td>242</td>
</tr>
</tbody>
</table>

6. Accuracy of the set zero-point determinations

This section studies further the quantitative values of the set zero-point (pseudo-) variances. First, there is an obvious variation with the scan geometry. Minimum variance occurs for RGC:s perpendicular to the ecliptic and maximum variance at minimum \(90^\circ - \xi\) inclination. As a useful approximation, we may write the standard deviation as

\[ \sigma = \sigma_0 (\Phi) \sigma_1 \cos 2\nu \quad (7) \]

where \(\nu\) is the "position angle" of the scan pole relative to the sun \((\nu = 0 \text{ or } 180^\circ \text{ for scan-pole in the ecliptic})\). Table 4 shows determinations of \(\sigma_0\) and \(\sigma_1\) for the "standard" configuration \((\nu = 0.9, 5 \text{ scans per set, 10000} \)
stars). The ratio \( \sigma_0/\sigma_1 \) should be less than 10% when all sets are included, and this modulation thus has little effect on the solution.

Table 4. Mean zero-point errors as function of the number of sets included.

<table>
<thead>
<tr>
<th>NSET</th>
<th>( N_s )</th>
<th>( \sigma_0 )</th>
<th>( \sigma_1 )</th>
<th>( \sigma_1/\sigma_0 )</th>
<th>( (\sigma_0)_{\text{calc}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>55.6</td>
<td>0.252</td>
<td>0.040</td>
<td>0.16</td>
<td>0.252</td>
</tr>
<tr>
<td>122</td>
<td>74.5</td>
<td>0.210</td>
<td>0.024</td>
<td>0.11</td>
<td>0.207</td>
</tr>
<tr>
<td>152</td>
<td>104.3</td>
<td>0.165</td>
<td>0.025</td>
<td>0.15</td>
<td>0.165</td>
</tr>
<tr>
<td>182</td>
<td>125.3</td>
<td>0.144</td>
<td>0.021</td>
<td>0.15</td>
<td>0.143</td>
</tr>
<tr>
<td>230</td>
<td>151.3</td>
<td>0.123</td>
<td>0.015</td>
<td>0.12</td>
<td>0.122</td>
</tr>
<tr>
<td>304</td>
<td>171.4</td>
<td>0.107</td>
<td>0.013</td>
<td>0.12</td>
<td>0.106</td>
</tr>
<tr>
<td>365</td>
<td>180.2</td>
<td>0.099</td>
<td>0.010</td>
<td>0.10</td>
<td>0.099</td>
</tr>
</tbody>
</table>

Also given in Table 4 is the effective number of observations per set, \( N_s \). This number has an upper limit about 200 (for 10000 stars) but is diminished by the increasing fraction of stars with fewer than 4 observations for diminishing NSET. The \( \sigma_0 \)-values are well reproduced by the simple expression

\[
(\sigma_0)_{\text{calc}} = 1.10 \left( \frac{\text{NSET}}{730} \right)^{-0.27} N_s^{-0.5}
\]  

as shown in the last column of Table 4. For a "real" solution with NSET = 730 we can thus extrapolate a set zero-point error \( \sigma_0 = 1.10 N_s^{-0.5} \) in units of the abscissa mean error. (This is satisfyingly close to the obvious assumption \( \sigma_0 = \sigma_N N_s^{-0.5} \)).

To make sure that the above estimate of the zero-point errors is not influenced by the simultaneous processing of stars with different abscissa mean errors, the following experiment was performed. With NSET = 182, \( N = 0^\circ 90 \), 5000 "bright" stars with abscissa error = 1 were combined with 20000 "faint" stars with abscissa error = 2 into a single set of normal equations. The resulting zero-point errors were virtually identical to those from a run with 10000 "bright" stars only. As expected thus, the total weight of a solution can be written

\[
W_{\text{eff}} = \sum \frac{N_k}{\sigma_k^2}
\]  

(9)
for a distribution where \( N_k \) stars have abscissa mean error \( \sigma_{uk} \). The total mean zero-point error is then

\[
\sigma_0 = 7.8 \left( \frac{w}{w_{\text{eff}}} \right)^{-0.5}
\]

(10)

for 1 year of observation with 5-scan sets.

For the "old" Step 2, the idea was to use about 1000 PRS-stars with \( \sigma_u \approx 3.0 \) mas which by (10) corresponds to \( \sigma_0 = 0.74 \) mas. This is about the set zero-point error contribution assumed in NDAC/LO/017, but with the "new" Step 2/3 there is a great improvement. As a rough example, we may use the data in NDAC/LO/017, supposing that 75\% of the stars brighter than 9th magnitude can ultimately be used in the zero-point solution. We find then

\[
w_{\text{eff}} = \frac{2250}{(2.99)^2} + \frac{4050}{(3.14)^2} + \frac{10400}{(3.38)^2} + \frac{30600}{(3.72)^2} = 3774 \text{ (mas}^{-2}\text{)}
\]

which gives \( \sigma_0 = 0.13 \) mas.

7. Conclusions

With no global parameters yet included, the Step 2/3-solution method given in Table 1 of WP 7100 seems to be quite feasible. Lindegren's method to compute the pseudo-solution and the pseudo-variances works well even though the normal equations are not quite rank-defect. As expected, the zero-point errors diminish by the inverse square root of the number of stars used, and the present Step 2/3-method is thus much superior to the old PRS-method. Further conclusions will be given in a second report studying the determinacy of the global parameters.