SMALL-SCALE EXPERIMENTS IN STEP 2 WITH ELIMINATION OF SET ORIENTATIONS
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1. Introduction
In order to study questions of rank-deficiency and solution methods, we have made some very small-scale numerical experiments on the HP 21MX computer at Lund Observatory. In all cases, the orientation parameters were eliminated to give 60x60 normal equations matrices (positions and parallaxes for 20 stars). Such matrices are easily stored in primary memory, thus non-sophisticated programs could be used. For details about the orientation elimination and normal equations formation we will refer to the note (Organisation of PRS experiments, 83-03-18) by Lindegren. After the change of elimination philosophy reported in NDAC/LO/020, the present experiments are partly irrelevant, but some of the results may still be useful.

2. Observation simulation
All simulations have been made with 20 "standard" stars, located near (~ 2° random deviations) the vertices of a regular dodecahedron. A full 2 ½ year mission was simulated (ξ = 43°, K = 6.4, cf Lindegren 82-05-19), but with a reduced number of sets and scans. One set was always assumed to include 5 scans of the satellite, but the duration of a set was scaled to the field width (w). With w in degrees, the duration of each set was 0.56 w days, and the total number of sets is then

NSET = 1630/w

(1)
Each scan was made with a fixed pole, and each star within \( \omega/2 \) of the scan
great circle was noted as "observed". For each star thus observed in at
least one of the five scans of the set, one observation equation of unit
weight was formed for the mean time of the set. (This equal weighting seems
justified from the data in NDAC/LO/017.) The observation equations can be
obtained from the formulae (12.5) and (12.6) in Annex A, using \( \lambda, \beta \) instead
of \( \alpha, \delta \). (In the parallax term, the sun is taken as the barycentre, thus
\( \beta_b = 0 \).) We find then for star \( i \) in set \( j \):

\[
\frac{1}{\cos \beta_i} \left( \frac{\partial u}{\partial \alpha} \right)_{ij} (\Delta \alpha \cos \beta)_i + \left( \frac{\partial u}{\partial \beta} \right)_{ij} \Delta \beta_i + \left( \frac{\partial u}{\partial \pi} \right)_{ij} \Delta \pi_i - c_j = \Delta u_{ij}
\]  

(2)

with coefficients

\[
\frac{1}{\cos \beta} \frac{\partial u}{\partial \alpha} = \sec^2 \rho \left[ \cos \beta \sin \beta_r - \sin \beta \cos \beta_r \cos(\lambda - \lambda_r) \right]
\]  

(3)

\[
\frac{\partial u}{\partial \beta} = \sec^2 \rho \left[ \cos \beta_r \sin(\lambda - \lambda_r) \right]
\]  

(4)

\[
\frac{\partial u}{\partial \pi} = R \sec^2 \rho \left[ \sin \beta \cos \beta_r \sin(\lambda_r - \lambda_b) + \sin \beta_r \cos \beta \sin(\lambda_b - \lambda) \right]
\]  

(5)

In these formulae, \( (\lambda, \beta) \) are the coordinates of the star, \( (\lambda_r, \beta_r) \) those of
the reference pole, and \( (\lambda_b, 0) \) those of the sun (with \( R \) the length of the
radius vector). The ordinate, \( \rho \), is given by

\[
\sin \rho = \sin \beta \sin \beta_r + \cos \beta \cos \beta_r \cos(\lambda - \lambda_r)
\]  

(6)

The observation equations (2) are the standard ones, with the abscissa
zero points \( c_j \) to be eliminated. As will be seen below, we were forced to
try also a more general expression, with $\Delta \lambda_1 \cos \beta_1$ and $\Delta \beta_1$ as additional unknowns. The modified observation equations are then

$$
\frac{1}{\cos \beta_1} \frac{\partial u}{\partial \lambda_1} (\Delta \lambda \cos \beta)_1 + \ldots + \frac{1}{\cos \beta_r} \frac{\partial u}{\partial \lambda_r} (\Delta \lambda \cos \beta)_r + \\
+ \frac{\partial u}{\partial \beta_r} (\Delta \beta)_r - c_j = \Delta u_{ij}
$$

(7)

with

$$
\frac{\partial u}{\partial \lambda_r} = -\frac{\partial v}{\partial \lambda}
$$

(8)

$$
\frac{\partial u}{\partial \beta_r} = -\tan \rho \sec \rho \cos \beta \sin(\lambda - \lambda_r)
$$

(9)

The latter equations follow from the fundamental equation for the abscissa

$$
\tan \nu = \frac{\sin \beta \cos \beta_r - \cos \beta \sin \beta_r \cos(\lambda - \lambda_r)}{\cos \beta \sin(\lambda - \lambda_r)}
$$

(10)

cf NDAC/LO/021, Eq (12).

3. Solutions from the standard observation equations

In the observation equations (2), the abscissa zero points $c_j$ were considered as "local" unknowns to be eliminated. The elimination was performed using (in principle) the formulae by Lindegren (83-03-18), and reduced sets of 60x60 normal equations were obtained. Contrary to our first expectations, these systems were not rank-deficient but yielded unique solutions. In order to study this further, we determined the eigenvalues for the normal equations matrices (see Fig 1). The largest eigenvalues are
generally about 20, and there is a definite "discontinuity" such that the three smallest ones fall significantly below the rest. For the range $w = 30^\circ$ to $w = 5^\circ$ we found empirically

$$\lambda_s \sim 3 \times 10^{-4} w^{2.6}$$

(11)

where $\lambda_s$ is the geometric mean of the three smallest eigenvalues. The determinacy of the normal equations is thus rapidly diminishing with diminishing $w$, and for small $w$ the system is practically rank-deficient.

These results can be understood as follows. The theoretical expectation of a rank-deficiency of three derives from the three degrees of freedom for relative position measurements. Any rotation of the coordinate system should give identical relative abscissae. This is only true when the ordinates are exactly zero, however, as is most easily seen when the pole of rotation is on a reference great circle. When the stars in a set are distributed in a broad band, and when the RGC pole $(\lambda, \beta)$ is fixed in a certain coordinate system, the relative abscissae are not independent of this system, and thus the normal equations will have a unique solution. The mean errors of the coordinates (as obtained from the inverse NE matrix) are approximately given by

$$\sigma^2_\lambda \sim 0.06 + 220 w^{-2.5}$$

$$\sigma^2_\beta \sim 0.03 + 280 w^{-2.5}$$

(12)

in units of the abscissa mean error. Comparing (11) and (12) the similar trend with $w$ is obvious.

The above solution is practically useless for realistic ($\leq 1^\circ$) $w$-values. The normal equations are then nearly rank-deficient, and some pseudo-solution must be obtained. In the present experiments we simply
"fixed" three position-values (usually \( \lambda, \beta \) for one star plus \( \beta \) for another) by adding large numbers to the corresponding diagonal elements in the NE matrix. This is equivalent to deleting the corresponding rows and columns of the matrix, and the coordinate-system is then anchored to the three fixed position-parameters. The mean errors are now instead

\[
\sigma_\lambda \sim 0.27 + 2.3/w
\]

\[
\sigma_\beta \sim 0.25 + 1.2/w
\]  

(13)

The increasing mean errors for smaller \( w \) are caused mainly by the increased ratio of the number of unknowns (NSET + 60) to the number of observations (\( \sim 620 \)) and the \( w^{-1} \)-dependence is most likely accidental. More significant is the similarity of Eqs (13) to the parallax error

\[
\sigma_\Pi \sim 0.34 + 1.4/w
\]  

(14)

as obtained from both the original and the "fixed" equations. (The parallaxes are "absolute", that is independent of the coordinate-system.) The deterioration of the original position-solution with decreasing \( w \) shown in (12) is thus more rapid than can be explained by the increased number of unknowns. This means that the position-system itself becomes less defined with decreasing \( w \), and supports our above explanation of the lack of rank-defect.

In the simulations, the right members of the observation equations were either random or equal to a theoretical (O-C) computed from (10) with true vs. a priori positions. The derived corrections in the random case were in good accord with the above mean errors, but in the (O-C)-case one has to be careful when "fixing" the normal equations. The (O-C)s were computed with non-zero "errors" for the fixed variables also, and the position
corrections for the remaining variables then become erroneous. (This was first thought to be a real problem, but it is now seen to be due to a conceptual error.)

4. Solutions with three orientation parameters

In order to check definitely the rank-deficiency, we made some further experiments with the observation equations (7). All three reference orientation parameters \( c_j, \Delta \lambda_{r_j}, \Delta \beta_{r_j} \) were now eliminated (cf Lindegren 83-03-18), and the eigenvalues of the resulting normal equations were again determined (Fig 2). In the interval \( \omega = 10^\circ \) to \( \omega = 40^\circ \), the three lowest eigenvalues are now approximately

\[
\lambda_s \approx 200 \omega^{-5}
\]  

(15)

This increase with decreasing \( \omega \) seems puzzling at first, but it is probably mostly a numerical effect. The important point is the low values for large \( \omega \), quite in contrast to Eq (11) above. For smaller \( \omega \)-values, there may be some effect that increases all eigenvalues by an absolutely small amount, which in this case has a large relative effect. (The elimination of three orientation parameters in each set requires the inversion of "local" 3x3 normal equations matrices from the orientation coefficients. As \( \omega \) diminishes, the determinacy of these matrices is greatly reduced, giving large numerical values in the inverses. The added numerical noise in the normal equations matrix may be sufficient to render it positive definite. For lack of computer capacity, all computations have been made with a 23 bit mantissa single precision.)

The (almost) rank-deficient normal equations were again solved by "fixing" three position parameters, and relations similar to Eqs (13) and (14) were found. The slope with \( \omega^{-1} \) is much steeper, however, and the solution is even weaker than estimated from the increased number of unknowns \( 3 \cdot \text{NSET} + 60 \).
5. Conclusions

The experiments reported here were made with only 20 stars with correspondingly large field-widths. No quantitative extrapolations to realistic cases \( N \geq 1000, w \sim 1^\circ \) should be attempted, but the following points can be made.

The lack of rigorous rank-deficiency for the normal equations formed by elimination of the abscissa zero-points was shown above to be due to geometrical effects of the finite field-width. For \( w \sim 1^\circ \), the equations should be nearly rank-deficient, however, and the pseudo-solution approach (NDAC/LO/018) should be readily applicable.

The somewhat artificial elimination of the RCC pole coordinates does produce a better rank-deficiency, but at the cost of numerical problems for small \( w \). The resulting normal equations also give larger position errors than the standard ones, even when corrected for the increased number of unknowns. It seems thus safe to conclude that the standard procedure to solve only for the abscissa zero-points is to be preferred. There is neither theoretical nor numerical justification for the (more laborious) elimination of three orientation parameters per set instead of one.
Fig. 1. Example plot of eigenvalues for normal equations where only one orientation parameter ($c_j$) has been eliminated from each set. Field width $w = 20^\circ$. 
Fig. 2. Example plot of eigenvalues for normal equations where three orientation parameters \((c_j, \Delta \lambda_{rj}, \Delta \beta_{rj})\) have been eliminated from each set. Field width \(w = 25^\circ\).