On Dynamical Smoothing

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1. Introduction

In this note I will give my interpretation of the concept 'Dynamical Smoothing' (DS) and indicate how it can perhaps be incorporated into the NDAC data processing. For simplicity, I will regard 'Numerical Smoothing' (NS) as a variant of DS, although there is a distinction to be explained later.

The purpose of DS is to enhance the astrometric accuracy by introducing dynamical (or numerical) constraints to the permitted attitude motion; i.e. by reducing the degrees of freedom of the attitude model. This is clearly possible only when the attitude can be modelised to mas accuracy with very few free model parameters per time interval. The attitude can then be interpolated so as to bridge the gaps between relatively bright stars which are normally not in the FOV together, and for which the photon noise is less. The abscissa solution will benefit in two ways from this: for the bright stars, the angles between them will be measured more accurately than is possible via intermediary faint stars; for the faint stars, on the other hand, the interpolated attitude provides a more accurate reference point than the frame attitude parameter ($\Delta \omega$) determined only from the (faint) stars in that specific frame.

A note of warning should be sounded here, however. Constraining the motion be means of a specific model obviously makes the solution more vulnerable to modelisation errors. With an unsuitable and slightly erroneous model there is a definite risk that systematic errors creep in which may be very difficult to detect and which may even in the end make the DS solution less accurate (if more precise) than the 'geometrical' solution.
In my opinion, the 'geometrical' solution (without DS) is therefore absolutely essential as a reference solution at least until the validity of the DS model has been thoroughly demonstrated in runs with actual data.

2. Formulae

Characteristic of the 'geometrical' solution is that the frame attitude parameter, $\Delta \omega = \Delta a_z \sin \delta_z + \Delta \omega$, is treated as one independent unknown per frame. With DS, $\Delta \overline{\omega}$ must be replaced by some function of time depending on a small number of attitude parameters common to a longer time interval, and these parameters must be determined by the set solution.

It would be tempting to substitute e.g. a polynomial for $\overline{\Delta \omega}(t)$ directly into the geometrical observation equation, using the polynomial coefficients as unknowns instead of the $\Delta \omega$'s. This would not be correct, however, since $\Delta a_z$ and $\Delta \omega$ are defined as corrections to the a priori attitude reconstituted from gyro and SM data. This reconstitution contains a number of dynamically impossible irregularities (on the mas level), e.g. the discontinuous third derivatives in the spline representation. Thus, even though $\overline{\Delta \omega}$ is forced to be a very smooth function, the resulting attitude, (a priori) + $\overline{\Delta \omega}$, is still dynamically impossible. This problem is however overcome with a simple trick.

Let $\tilde{a}(t) = (\tilde{a}_z, \tilde{\delta}_z, \tilde{\omega})'$ be the a priori attitude obtained from the attitude reconstitution, and $\overline{d} = (\sin \delta_z', 0, 1)'$. Then

$$\overline{\Delta \omega}(t) = \overline{d}'(\overline{a} - \tilde{a}) = (a_z - \tilde{a}_z) \sin \delta_z + (\omega - \tilde{\omega})$$

(1)

Now let the attitude model be a function depending on some parameter vector $\overline{h}$:

$$\overline{a} = \overline{a}(t; \overline{h})$$

(2)

This means that $\overline{a}(t)$ is completely determined by the parameters $\overline{h}$. For a truly dynamical model, $\overline{a}(t; \overline{h})$ must be a solution of
the Equations of Motion (EM), with \( h \) containing the initial values (or equivalent) and a complete parametrization of the external torques, as required for a unique solution.

A disadvantage with the true DS is of course that a rigorous solution to the EM cannot in general be obtained in closed form; thus analytical approximations or numerical integrations are required. Numerical Smoothing (NS) sidesteps this by expressing \( \hat{a}(t; h) \) directly in terms of simple analytical functions, e.g. polynomials and trigonometric series. It must however be able to reproduce any reasonable solution of the EM. Thus, if \( \{a\}_\text{EM} \) is the set of motions permitted by the EM and \( \{a\}_\text{NS} \) the set of motions permitted by the NS model, we must have

\[
\{a\}_\text{EM} \subseteq \{a\}_\text{NS}
\]  

If this is satisfied, it is however quite probable that \( \{a\}_\text{NS} \) is much wider than \( \{a\}_\text{EM} \), in which case the NS is clearly inferior to DS. Only if the NS model can be narrowed down almost to equality in (3) will NS be as performant as true DS. This may be possible by using analytical approximations to the EM as the starting point for formulating the NS model, in which case the distinction between NS and DS really disappears.

Returning to Eqs (1) - (2), we must now introduce \( \Delta h \) as unknowns in the set solution instead of \( \Delta \hat{w} \). This requires that \( \hat{a}(t; h) \) is linearized about some 'permitted' motion \( \hat{a}(t; h_0) \) close enough to the true motion. It is natural to choose \( h_0 \) such that \( \hat{a}(t; h_0) \) approximates \( \hat{a}(t) \) in a least-squares sense; the choice is however uncritical. With

\[
\Delta h = h - h_0
\]

and

\[
L = \frac{\partial \hat{a}}{\partial h} \bigg|_{h_0} = \begin{pmatrix}
\partial \omega_2 / \partial h_1 & \partial \omega_2 / \partial h_2 & \ldots \\
\partial \delta_2 / \partial h_1 & \partial \delta_2 / \partial h_2 & \ldots \\
\partial \omega / \partial h_1 & \partial \omega / \partial h_2 & \ldots
\end{pmatrix}
\]

we can now write (1) as
\[ \bar{\omega}(t) = \bar{d}'[\tilde{a}(t; h_0) + L(t) \Delta h - \tilde{a}(t)] \]

= \bar{d}'L(t) \Delta h + \bar{d}'[\tilde{a}(t; h_0) - \tilde{a}(t)]

= \Sigma_k \left( \frac{\partial \alpha_z}{\partial h_k} \sin \alpha_z + \frac{\partial \omega}{\partial h_k} \right) \Delta h_k +

+ [\alpha_z(t; h_0) - \tilde{\alpha}_z(t)] \sin \alpha_z + [\omega(t; h_0) - \tilde{\omega}(t)] \]  \hspace{1cm} (6)

The last two terms are computable from the \textit{a priori} attitude and can be transferred to the right-hand side of the observation equation. This is of course equivalent to replacing \tilde{\tilde{a}}(t) by \tilde{a}(t; h_0) when computing \mathbb{G}_{\text{cat}}, which is another possibility. The important result is however that \bar{\omega} has been replaced by the unknowns \Delta h_k factored as in (6).

3. Implications for the Set Solution

It is seen that DS is formally a rather trivial modification of the observation equations of a set. It does not affect other parts of the data processing, which is natural since the 'dynamical memory' of the satellite cannot be as long as 12 h.

Computationally, the implications for the Set Solution may be dramatic, however. This depends on the filling in of the normal equations matrix when eliminating the \Delta h_k's. In this process, stars much further apart than the FOV width will be linked, producing a non-zero element at the corresponding place in the abscissa matrix. The width of the strip being filled in depends on the support of the partial derivatives in (6), i.e. on the interval in which they are non-zero. If true DS is employed over the whole interval between jet firings, then the support will equal that interval of perhaps many minutes of time. This will be the case even if the 'dynamical memory' has faded considerably over the interval. Certain analytical representations (splines in particular) will on the other hand have limited support, but are of course not as good as true DS. Clearly, various degrees of DS become possible with certain NS techniques, and a compromise against required computing time may be sought.