Space Astrometry (HIPPARCOS)

HOW MUCH WEIGHT CAN BE GAINED ON HIGH-PRIORITY STARS?

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SUMMARY

From simulated observations of 50 stars along a great circle we have derived the positional accuracy as a function of magnitude and accumulated observing time. The experiments indicate that the mean error of a high-priority star can rarely be reduced by more than 10 %, by giving it proportionally longer observing time at the level of IFOV switching. In our opinion it is doubtful whether a general priority rating of programme stars is worth while.

1. INTRODUCTION

For the majority of stars observed by HIPPARCOS, the accuracy will depend mainly on magnitude and position (especially ecliptic latitude), cf Table 5.2-5.3 in the Phase A Report SCI(79)10, p 71. There may be, however, a small number of high-priority stars for which enhanced accuracy would be particularly valuable from a scientific point of view, even though this must be gained at the expense of a slight degradation of the results for the other stars, or a more substantial degradation for a number of low-priority stars. It is of considerable interest to know how much relative improvement is possible for such high-priority stars.

The accuracy of the results for a given star can be increased only by increasing the accumulated observing time on the star. This can be achieved in two ways: either by observing it for a longer time each time the star is being scanned, or by increasing the number of such scans across the star. In the first case, which is the subject of the present study, there is a redistribution of observing time according to priority, effected by the IFOV switching strategy. In the second case, the nominal scanning law is occasionally modified by suitable attitude manoeuvres. While this may be quite an efficient way to increase the weight of certain small areas of the sky, it obviously cannot be applied to the observation of a rather large number of high-priority stars dispersed over the whole sky. The method is therefore not further discussed in this note.
2. EXPERIMENTS

Observations of \( n = 50 \) stars randomly distributed along a great circle were simulated, using two fields of view (POV) separated by 68.5°, each of length 14.4°. The average number of stars in the field is then \( \mu_0 = 4.0 \), close to the value for the real mission (for which \( n = 800 \) however). Let \( \alpha_i \) be the abscissa of the \( i \)-th star \( (i = 1, 2, \ldots, n) \) and \( \omega_j \) the attitude argument at the midpoint of the \( j \)-th frame \( (j = 1, 2, \ldots, M) \). In differential form the observation equations are

\[
\Delta\alpha_i - \Delta\omega_j = \Delta\eta_{ij} + \varepsilon_{ij}, \tag{1}
\]

where \( \eta_{ij} \) is the field coordinate along the scan of star number \( i \) as measured in frame \( j \), and \( \varepsilon_{ij} \) is the estimated mean error of the measurement. The arguments of successive frames differ by 1.44°, corresponding to a frame overlap factor \( Q_f = 0.9 \). This means that a given star is visible throughout nine consecutive frames, and that the average number of observations per frame is \( <m> = Q_f c_m = 3.6 \). Complete scans of 360° were generated, comprising \( M = 250 \) frames and about 900 observations of the form (1). The right members \( \Delta\eta_{ij} \) were generated by means of the random generator GAUSS with standard deviation \( \varepsilon_{ij} \). The normal equations were solved subject to the constraint

\[
\sum_i \Delta\alpha_i = 0. \tag{2}
\]

Before forming the normal equations, the attitude arguments \( \omega_j \) were eliminated as described in Appendix B.

Two catalogues A and B were created, each containing the abscissae \( (\alpha_i) \), magnitudes \( (H_i) \), and priorities \( (P_i) \) of 50 stars:

\[
\alpha_i = (360°/n)(i - 1) + a \times \text{RAND} \quad (\text{mod} \ 360°), \quad a = 14.4°; \tag{2a}
\]

\[
H_i = 8.7 + 1.3 \times \text{GAUSS}; \tag{2b}
\]

\[
P_i = \text{INT}(4.5 + 1.5 \times \text{GAUSS}), \quad 1 \leq P_i \leq 7; \quad i = 1, 2, \ldots, n = 50. \tag{2c}
\]

RAND and GAUSS are pseudo-random numbers with uniform \( [0,1] \) and normal \( N(0,1) \) distributions, respectively (Lindegren, 1980-10-20). (2a) gives a slightly more regular distribution of abscissae than a uniform and independent distribution on \( [0,360°] \), mimicking perhaps the selection of real stars to make a reasonable regular mesh. This is reflected in the distribution of the number of frames with \( m = 0, 1, 2, \ldots \) observable stars (Table 1), which is more peaked than the corresponding Poisson distribution. The distributions of \( H_i \) and \( P_i \) are shown in Table 2.

The mean errors \( \varepsilon_{ij} \) follow from the magnitudes \( H_i \) and observing times \( t_{ij} \). Since the duration of a frame was set to 2 sec, we have \( \varepsilon_i t_{ij} = 2 \) sec for all \( j \). Conforming with the time hierarchy proposed in a previous note (Lindegren, 1980-02-22), observing time was allotted in multiples of 0.2 sec, presently referred to as shares. If a star receives \( k_i \) shares, we have put

\[
\varepsilon_{ij} = \sigma_{18}^{-1} H_i (0.2 k_i)^{-\frac{1}{4}}, \tag{3}
\]

where \( \sigma_{18} \) is the mean error for 1 sec integration time as function of magnitude,

\[
\sigma_{18}(H) = (3.8 + 81300 \frac{1 + 29/f_i}{f_i})^{\frac{1}{4}} \text{ mas}; \tag{4a}
\]

\[
f_i = 1780 \times 10^{-0.4(H-9)} \text{ Hz} \tag{4b}
\]
Table 1. Distribution of $m = \text{number of observable stars in a frame (2 sec interval)}$. The Poisson distribution with the same mean is shown for comparison.

<table>
<thead>
<tr>
<th>No. of observable stars ($m$)</th>
<th>No. of frames with exactly $m$ stars</th>
<th>Poisson</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>7.1</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>25.2</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>45.0</td>
</tr>
<tr>
<td>3</td>
<td>74</td>
<td>53.4</td>
</tr>
<tr>
<td>4</td>
<td>99</td>
<td>47.6</td>
</tr>
<tr>
<td>5</td>
<td>39</td>
<td>33.9</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>20.2</td>
</tr>
<tr>
<td>$\geq 7$</td>
<td>0</td>
<td>17.6</td>
</tr>
<tr>
<td>total</td>
<td>250</td>
<td>250.0</td>
</tr>
<tr>
<td>$&lt; m \geq 7$</td>
<td>3.564</td>
<td>3.564</td>
</tr>
</tbody>
</table>

Table 2. Distribution of magnitudes ($H$) and priorities ($P$) for the union of the two catalogues A and B (100 stars in total).

<table>
<thead>
<tr>
<th>$H$</th>
<th>$P$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>any $P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 6.5$</td>
<td></td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>17</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>13</td>
<td>1</td>
<td>0</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>0</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>$&gt;11.5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>any $H$</th>
<th>6</th>
<th>9</th>
<th>24</th>
<th>15</th>
<th>31</th>
<th>11</th>
<th>4</th>
<th>100</th>
</tr>
</thead>
</table>
Shares were distributed by means of an algorithm described in Appendix A. The main features are as follows. Given $m < 10$ stars observable in a frame, the distribution of the ten shares is such that

(a) every star receives at least one share;
(b) fainter stars receive if possible more shares than bright stars;
(c) stars of higher priority receive if possible more shares than those of lower priority, provided $\text{AMP} > 0$ and $m > 2$.

AMP is a non-negative parameter determining the degree to which the priority rating of the stars is taken into account. The rating is without effect if $\text{AMP} = 0$. It is also effectively neglected when $m = 1$ (in which case the one visible star receives all ten shares) or $m = 2$ (in which case the shares are divided approximately in proportion to $\sigma_1^2(h_1)$, thereby minimising the error in the angle between the two stars).

For each of the catalogues A and B, five solutions were made with $\text{AMP} = 0, 0.5, 1, 2$, and 4. It should be noted that the observation equations are identical in each series of five solutions, except for the weights $\varepsilon_{ij}^{-2}$. The results discussed in the following section are the abscissa mean errors ($\sigma_i$), obtained from the covariance matrix, and the accumulated number of shares given to the different stars ($K_i$). The actual solutions for the abscissa corrections were only used for checking purposes.

3. RESULTS

Fig. 1 gives an example of the redistribution of shares when increasing the parameter AMP from 0 to 1 and 4. The relative variation of abscissa mean errors when going from $\text{AMP} = 0$ to AMP = 0.5, 1, 2, and 4 is shown for both catalogues in Figs 2a - d. The mean errors of the high-priority stars are never reduced by more than $16\%$ and are in fact often increased, especially for relatively bright stars.

Figs 3a - e show the mean errors ($\sigma_i$) versus accumulated observing time (in shares; $K_i$) for all ten experiments, grouped according to magnitude. The function

$$\sigma(K) \equiv (\sigma_\infty^2 + c \sigma_{1s}^2(H)/(0.2K))^{1/2},$$

(5)

depending on the two parameters $\sigma_\infty$ and $c$, was fitted to the points in each of Figs 3a - e. The results, Table 3, show that the two parameters are very nearly independent of magnitude; moreover, the dimensionless number $c$ is not significantly different from unity. The existence of an asymptotic error $\sigma_\infty$, not very much smaller than the average mean error $<\sigma_i> \approx 3.1$ mas, explains why the allotment of extra observing time has so little effect on the accuracy, especially for bright stars.

By means of (5) it is easily shown that the maximum possible reduction of $\sigma_i$, viz. when increasing the observing time from its average value (depending on magnitude, but typically 10 sec) to the theoretical maximum $32.4$ sec ($162$ shares = 9 shares in each of 18 frames), is about $20\%$. The result can be extrapolated to the full-scale solution ($n = 800$), since there are good reasons to assume that the greater number of stars (keeping $<m>$ constant) merely increases the asymptotic mean error. According to Høyer, Poder, Høg, and Lindegren (work in progress, 1980), the average abscissa variance is only slightly underestimated by the formula.
\[
\langle \sigma^2_i \rangle = \frac{1}{(1 - Q_f)} \left[ 1 + \frac{n^{1/3}}{2(<m>-1)} \right] \langle \varepsilon_{ij}^2 \rangle.
\] (6)

In the present case, \( n = 50, <m> = 3.6, Q_f = 0.9 \), and \((\sigma_i)_{\text{rms}} = 9.5 \) mas, we find \((\sigma_i)_{\text{rms}} = 2.8 \) mas from the formula (experimental value = 3.1 mas). With \( n = 800 \) the rms mean error should increase by a factor 1.28, according to (6), indicating an asymptotic mean error of \( \sigma_\infty = 3.3 \) mas for the full-scale problem. The corresponding mean errors with average and maximum observing time are summarised in Table 4.

Allocating maximum possible observing time to a given star, without causing a general weakening of the solution, is probably a rather idealized situation, requiring for instance that there are only few and bright stars competing for time. Therefore it is rather unrealistic to expect more than about 10% improvement of the mean error for a high-priority star. In view of the comparable size of the scatter of individual mean errors (cf Figs 3a–e) and the much larger variations depending on position and other factors, we believe that it is hardly worth while to make a general priority rating of programme stars.

Table 3. Least-squares solutions for the asymptotic mean error \( \sigma_\infty \) and coefficient \( c \) in (5), cf Figs 3a–e.

<table>
<thead>
<tr>
<th>H</th>
<th>( \sigma_\infty ) (mas)</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>2.19 ( \pm ) 0.06</td>
<td>0.85 ( \pm ) 0.07</td>
</tr>
<tr>
<td>8</td>
<td>2.21 ( \pm ) 0.08</td>
<td>0.87</td>
</tr>
<tr>
<td>9</td>
<td>2.31 ( \pm ) 0.10</td>
<td>0.93</td>
</tr>
<tr>
<td>10</td>
<td>1.99 ( \pm ) 0.14</td>
<td>1.02</td>
</tr>
<tr>
<td>11</td>
<td>2.55 ( \pm ) 0.47</td>
<td>0.83 ( \pm ) 0.07</td>
</tr>
</tbody>
</table>

mean | 2.20 | 0.98 |

Table 4. Predicted mean errors for the full-scale solution (800 stars) as functions of magnitude, assuming average and maximum \((t_{\text{max}} = 32.4 \) sec) integration time.

| H | \( t_{\text{ave}} \) (sec) | \( \sigma_i(H) \) (mas) | \( \sigma_i(t_{\text{ave}}) \) (mas) | \( \sigma_i(t_{\text{max}}) \) (mas) | improvement (%) |
|---|------------------|-----------------|-------------------|-------------------|----------------|---|
| 7 | 5.7 | 3.3 | 3.6 | 3.3 | 8 |
| 8 | 8.0 | 4.7 | 3.7 | 3.4 | 8 |
| 9 | 9.9 | 7.1 | 4.0 | 3.5 | 12 |
| 10 | 13.8 | 11.1 | 4.4 | 3.8 | 14 |
| 11 | 17.1 | 17.9 | 5.4 | 4.5 | 16 |
APPENDIX A

An algorithm for allotting shares of observing time to the stars in a frame, depending on their magnitudes and priorities

Common data (stars are identified by index i):

\( H(i) \) = HIPPARCOS magnitude, uniquely related to the weight per unit integration time of grid coordinate measurements, \( = \frac{1}{2}(B+V) \)

\( P(i) \) = priority, on an integer scale from 1 to 7; average = 4

External function:

\( \sigma_{ls}(H) \) = m.e. of field coordinate \( \eta \) for 1 sec integration time, eq (4)

Input data:

\( m \) = number of observable stars, i.e. stars within the FOV for the entire duration of the frame (2 sec)

\( ID(j) \) = list of observable stars; \( j = 1, 2, \ldots, m \)

\( NT \) = number of shares to be distributed (= 10 in present case)

\( AMP \) = amplification factor for priority rating (\( \geq 0 \))

Output data:

\( ID(j) \) = list of observable stars (same as input list, but generally in a different order)

\( IT(j) \) = number of shares allotted to star \( ID(j) \); \( \sum_{j} IT(j) = NT \)

Algorithm

if \( (m = 1) \) : \( IT(1) = NT \); end;

if \( (m \geq NT) \) : sort \( ID(j) \) according to non-decreasing \( P(ID(j)) \);

\[ IT(j) = 1, \quad j = 1 \rightarrow NT; \]

\[ IT(j) = 0, \quad j = NT+1 \rightarrow m; \] end;

else :

\[ c(j) = \sigma_{ls}(H(ID(j))) \times 1.5854 \times [AMP \times (m-2)/(m-1) \times (P(ID(j)) - 4)], \quad j = 1 \rightarrow m; \]

\[ C = c(1) + c(2) + \ldots + c(m); \]

sort \( ID(j) \) and \( c(j) \) according to non-decreasing \( c(j) \);

\( N = NT; \)

\[ IT(j) = \text{INT}(N \times c(j)/C + 0.5), \]

if \( (IT(j) < 1) \) : \( IT(j) = 1, \)

\[ N = N - IT(j), \]

\[ C = C - c(j), \]

end.
APPENDIX B

Elimination of the attitude argument from the observation equations

The observation equations (1) contain \( n + M \) unknowns, i.e. \( n = 50 \) abscissa corrections and \( M = 250 \) attitude arguments. Since we are in fact only interested in the abscissa corrections, it would be a good thing if the other unknowns could be eliminated at an early stage. This is possible with a variant of the Schreiber-Helmert Method (Poder, Hög, and Høy, Processing of Data from the Astrometry Satellite, 1979-08-10).

The observations equations of unit weight are of the form

\[ Ax + By = h, \]  
(\text{B.1})

where \( x \) is the \( n \)-vector of abscissa corrections, \( y \) the \( M \)-vector of attitude arguments, \( h \) the \( K \)-vector (say) of right-hand members. The normal equations are

\[
\begin{align*}
A^T Ax + A^T By &= A^T h, \\
B^T Ax + B^T By &= B^T h. 
\end{align*}
\]  
(\text{B.2})

Eliminating \( y \) we obtain the reduced normal equations

\[
[A^T A - A^T B (B^T B)^{-1} B^T A] x = A^T h - A^T B (B^T B)^{-1} B^T h. 
\]  
(\text{B.3})

But it is easily verified that the modified observation equations,

\[
[A - B (B^T B)^{-1} B^T A] x = h, 
\]  
(\text{B.4})

yield precisely equations (B.3), when normal equations are formed in the usual manner. In this respect, the modified observation equations (B.4) are equivalent to the original form (B.1).

In our particular case the elements of \( A \) and \( B \) are

\[
A_{ki} = \varepsilon^{-1}_{i,j(k)} \delta_{i,j(k)}, \quad B_{kj} = -\varepsilon^{-1}_{i(k),j} \delta_{j,j(k)},
\]  
(\text{B.5})

if \( i(k) \) is the star and \( j(k) \) the frame associated with observation number \( k \), \( k = 1, 2, \ldots, K \). \( B^T B \) is diagonal and positive definite:

\[
(B^T B)_{jj} = \sum_{k: j = j(k)} \varepsilon^{-2}_{i(k),j}.
\]  
(\text{B.6})

After multiplication by \( \varepsilon_{ij} \), the explicit form of the modified observation equation is

\[
(1 - w_{ij}) \Delta \alpha_i - \sum_{i' \neq i} w_{i',j} \Delta \alpha_{i'} = \Delta \eta_{ij} + \varepsilon_{ij},
\]  
(\text{B.7})

where

\[
w_{ij} = \frac{\varepsilon^{-2}_{ij}}{\sum_{i' \neq i} \varepsilon^{-2}_{i',j}}
\]  
(\text{B.8})

is the fractional weight of \( \Delta \eta_{ij} \) within its frame (the sums are taken over stars \( i' \) observed in frame \( j \)).
Fig 1. Examples of the redistribution of observing time as controlled by the priority amplification parameter AMP. The left diagram shows the accumulated observing time (in shares, one share = 0.2 sec) for AMP = 0 versus that for AMP = 1, while the right diagram shows the corresponding redistribution from AMP = 0 to AMP = 4. AMP = 0 means that the priority rating of the stars (P, shown in the figure as integers on a scale 1 to 7) is effectively ignored, while for increasing AMP the rating is increasingly taken into account when allotting observing time to the different stars.
Fig 2a-d. Relative change of abscissa mean errors ($\sigma_i$) when increasing the parameter AMP from 0 to 0.5, 1, 2, and 4, plotted versus priority ($P_i$). Results from both catalogues are plotted together.
Fig 3a–e. Abscissa mean errors as function of accumulated observing time (number of shares), for magnitude $H = 7, 8, 9, 10, \text{ and } 11 \ (\pm 0.5)$. The curves are the fitted relations (5) with parameters $\sigma_\infty$ and $c$ as in Table 3.
like P over the sky. But for perhaps 3 or 4 regions it appears quite feasible to double or triple the number of scans in this way. Since the results of different scans are probably rather independent, the mean errors in such regions could be reduced almost by a factor $\sqrt{2}$ to $\sqrt{3}$.

Fig 1a. Nominal revolving scanning. $z'z''$ is the path on the sky described by the nominal spin axis ($z$), which is at a nearly constant angle $\xi = 40^0$ from the sun (S). $p'p''$ is a great circle with one pole at P. P will be scanned whenever $z$ is about $90^0$ from P, i.e. at $z_1$, $z_2$, $z_3$.

Fig 1b. Modified scanning law giving a larger number of scans across P. The speed of $z$ along its path should be kept approximately constant, about 0.50° per scan, also during the special manoeuvres at $z_1$, $z_2$, $z_3$. 