Formulae for the preliminary design of Schmidt-Cassegrain systems

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1. Introduction

This note summarizes the formulae needed for a preliminary design of a Schmidt-Cassegrain system, i.e., an astronomical telescope consisting of a large concave primary mirror $M_1$ (Fig. 1), a smaller convex secondary mirror $M_2$, and a (transmitting or reflecting) corrector plate $P$. The preliminary design includes:

- computation of Gaussian (paraxial) parameters, i.e.,
  separation of elements, radii of curvature, equivalent focal length, and also obstruction ratio and field curvature, and

- computation of the fourth-order (aspheric) deformations of $M_1$, $M_2$, and $P$ for elimination of the first three Seidel aberrations (spherical aberration, coma, and astigmatism).

The remaining two third-order (Seidel) aberrations of a symmetrical instrument, i.e., field curvature and distortion, do not affect the quality of a point image (if a curved focal surface is admitted) and are therefore tolerable.

The detailed design of a Schmidt-Cassegrain system should of course take into account also higher-order aberrations and the effects of the asymmetric pupil and inclined correcting mirror. This can only be made by means of ray tracing and is beyond the scope of this study.

However, we shall be able to demonstrate, by an example, that already the preliminary design, as described in this note, can yield an instrument with a theoretical performance sufficiently good for the HIPPARCOS project.
Fig. 1. Definition of main parameters of a Schmidt-Cassegrain system with non-inclined zero-power corrector plate. Notations (all quantities positive):

- \( P \) = Schmidt corrector plate (paraxially flat)
- \( M_1 \) = concave primary mirror, radius of curvature \( r_1 = 2f_1 \)
- \( M_2 \) = convex secondary mirror, radius of curvature \( r_2 = 2f_2 \)
- \( h_0, h_1, h_2 \) = semidiameters of \( P, M_1, \) and \( M_2 \)
- \( x_1 \) = separation between \( P \) and \( M_1 \)
- \( d \) = separation between \( M_2 \) and \( M_1 \)
- \( \Delta \) = back focal length
- \( q \) = minimum obstruction ratio (zero unvignetted field)
- \( \eta = h_2/h_0 \) = practical obstruction ratio
- \( f_e \) = equivalent focal length.

2. Gaussian properties, dimensions, obscuration ratios, field curvature

Consider first the system with a paraxially flat (zero-power), non-inclined corrector plate (Fig. 1). The equivalent focal length is

\[
f_e = \frac{f_1 f_2}{d + f_2 - f_1}.
\]

The location of the focal surface is given by

\[
\Delta = \frac{f_2(f_1 - d)}{d + f_2 - f_1} = d
\]

and the minimum obstruction ratio by

\[
q = 1 - d/f_1.
\]
However, to allow an unvignetted field of angular semidiameter $\alpha$, the dimension of the secondary must be increased from $ch_0$ to $h_2 = \eta h_0$, where the practical obstruction ratio $\eta$ is

$$\eta = q + (d + qx_1) \tan \alpha / h_0$$  \hspace{1cm} (4)

The dimension of the primary mirror must similarly be increased from $h_0$ to

$$h_1 = h_0 + x_1 \tan \alpha.$$  \hspace{1cm} (5)

For an anastigmatic system the radius of curvature of the best focal surface equals the Petzval curvature radius

$$\rho = (1/f_2 - 1/f_1)^{-1}.$$  \hspace{1cm} (6)

In practice it will be necessary to give the corrector plate a small power, i.e., some paraxial curvature (concave), in order to minimize the maximum deviation of the plate from a flat surface. Let $r_0 = 2f_0$ be the radius of curvature of the (reflecting) corrector plate (Fig. 2). The corrector/primary now constitutes a system with equivalent focal length

$$f'_1 = \frac{f_0 f_1}{f_0 + f_1 - x_1}.$$  \hspace{1cm} (7)

After the primary mirror the Gaussian properties of the system are thus exactly the same as for a system with paraxially flat corrector and a

\[\text{Fig. 2. Definition of modified primary focal length } f'_1 \text{ for system with curved corrector (radius of curvature } 2f_0).\]
primary mirror $M'_1$ of focal length $f'_1$ positioned at

$$x'_1 = x_1 + \frac{f'_1 x_1}{f'_0 + f'_1 - x'_1}. \tag{8}$$

Thus, eqs. (1) and (2) remain valid provided $f_1$ and $x_1$ are replaced by $f'_1, x'_1$. Also eqs. (3) – (6) remain approximately valid (to sufficient accuracy for the present purpose) with these modifications. Moreover, since these modifications are of a second-order nature, they leave the third-order aberrations on the Petzval surface unchanged (cf. Ch. IV § 1.4 in Linfoot, Recent Advances in Optics, Oxford 1955). Consequently, the modified system with paraxially flat corrector can be taken as basis for computing the mirror deformations required to eliminate third-order aberrations.

3. **Definition of deformation coefficients**

Consider first a spherical primary mirror with radius of curvature $2f_1$. The surface is

$$x - x_1 = (4f_1^2 - y^2)^{\frac{1}{2}} - 2f_1$$

$$= - \frac{y^2}{4f_1} - \frac{y^4}{64f_1^3} - O(y^6) \tag{9}$$

For an aspheric primary, a dimensionless deformation coefficient $b_1$ is defined such that $b_1 = 0$ for a spherical mirror and $b_1 = -1$ for a paraboloid:

$$x - x_1 = - \frac{y^2}{4f_1} - (1 + b_1) \frac{y^4}{64f_1^3} - O(y^6). \tag{10}$$

Notice that the asphericity causes a retardation (increase of optical path length) by $- b_1 y^4/32f_1^3$.

Similarly, an aspheric secondary is described to fourth order by

$$x - (x_1 - d) = - \frac{y^2}{4f_2} - (1 + b_2) \frac{y^4}{64f_2^3} - O(y^6). \tag{11}$$

The retardation is in this case $+ b_2 y^4/32f_2^3$.

* Since the positions of the secondary and the focus are defined relative the primary, $d$ and $\Delta$ must be modified to refer to $M'_1$. 
The deformation of the Schmidt corrector normal to the optical axis is given by the coefficient $k$:

$$x = x^2/4f_o + ky^4.$$  \hspace{1cm} (12)

The retardation (relative to the spherical corrector) is $-2k_y^4$. The curvature of the corrector can be used to minimize the maximum thickness of the total deformation (12). This is achieved by having $x = 0$ at the edge of the corrector ($y = h_o$). The solution for $f_o$ is

$$f_o = -1/4k_h^2,$$  \hspace{1cm} (13)

and the maximum deviation from a plane is obtained at $y = \pm h_o/\sqrt{2}$, viz.

$$x_{\text{max}} = -k_h^4/4.$$  \hspace{1cm} (14)

Numerically, this is exactly one fourth of the maximum deformation needed with a paraxially flat corrector ($f_o = \infty$).

In reality the Schmidt corrector mirror must be inclined by some angle $i$ to the optical axis. The correct figuring of the initially flat surface should then be such as to produce the same retardation versus $y$ as eq. (12) for the non-inclined corrector. Referring to Fig. 3 we see that the figuring normal to the surface should be

$$t(y) = \frac{1}{\cos i} (y^2/4f_o + ky^4).$$  \hspace{1cm} (15)

![Figure 3](image_url)

**Fig. 3.** Retardation by deformation on inclined corrector. With deformation $t$ normal to the reference flat surface, the retardation is $-\vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{c} = -t \frac{\cos 2i}{\cos i} - t/\cos i = -2t \cos i$. 

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### Notes

1. The expression for the retardation $-2k_y^4$ assumes that the deformation $k_y$ is proportional to the square of the distance from the optical axis. This is a simplification that works well for small deformations.
2. The solution for $f_o$ is derived by setting $x = 0$ at the edge of the corrector ($y = h_o$) and solving for $f_o$.
3. The expression for the maximum deviation $x_{\text{max}} = -k_h^4/4$ provides a practical way to estimate the maximum thickness of the corrector.
4. The figuring normal to the surface in an inclined corrector is given by the equation $t(y) = \frac{1}{\cos i} (y^2/4f_o + ky^4)$, which takes into account the inclination of the corrector mirror.
The maximum thickness of the Schmidt deformation is therefore

\[ T_0 = -Y h_0^4 / 4 \cos i. \]  

(16)

(Note that \( i \) equals one fourth of the basic angle.)

The maximum fourth-order deformation of the primary and secondary mirrors is obtained at the extreme edges. However, by making reference to a spherical surface with a curvature slightly different from the paraxial curvature, the maximum deformation can be reduced by a factor four, viz.

\[ T_i = b_i h_i^4 / 256 f_i^3, \quad i = 1, 2. \]  

(17)

4. Computation of deformation coefficients

The fourth-order deformations of \( P, M_1 \) and \( M_2 \) needed to eliminate spherical aberration, coma, and astigmatism of a symmetrical Schmidt-Cassegrain system are easily computed from the Gaussian parameters \( f_1, f_2, d, \) and \( x_1 \). Following Linfoot (op. cit., Ch. IV § 2), the figuring depths \( A, B, \Gamma \) are obtained by solving the three linear equations

\[
\begin{align*}
A + B + \Gamma &= 1 - u \quad \text{(spherical aberration)} \\
3B + \sigma \Gamma &= 2 - uv \quad \text{(coma)} \\
3B + \sigma^2 \Gamma &= 4 - uv^2 \quad \text{(astigmatism)}
\end{align*}
\]  

(18)

where

\[
\begin{align*}
u &= q^2 (2\xi - q)^2 / \xi^3, \quad \xi = f_2 / f_1, \\
v &= 1 + 1 / (2\xi - q), \\
s &= 1 - 1 / q, \\
\sigma &= x_1 / f_1.
\end{align*}
\]  

(19)
The figuring depths are related to the previously defined deformation coefficients by the equations

\[ b_1 = -A, \quad b_2 = \frac{B}{2} / \epsilon^4, \quad \gamma = -\Gamma/64 f_1^3. \]  

(20)

5. Some numerical examples

5.1. The nominal HIPPARCOS instrument

The nominal instrument is defined e.g. in Section 3.7.1 of the ESTEC Technical Report on the extended phase A study. Primary characteristics are:

\[ r_1 = 2.044 \]
\[ r_2 = 1.102 \]
\[ d = 0.700 \]
\[ x_1 = 1.500 \]
\[ h_0 = 0.160 \]
\[ i = 18^\circ \]
\[ \alpha = 0.64^0 \]

paraxially flat corrector \((f_0 = \infty)\).

From these data we compute successively:

\[
\begin{align*}
 f_1 & = 1.022000000 \\
 f_2 & = 0.551000000 \\
 f & = 2.459048035 \\
 \Delta & = 0.074768559 \\
 q & = 0.315068493 \\
 \eta & = 0.393 \\
 h_1 & = 0.175 \\
 h_2 & = 0.063 \\
 \rho & = 1.195588110 \\
 \xi & = 0.539138943 \\
 u & = 0.568973568 \\
 v & = 2.310256410 \\
 s & = -2.173913043 \\
 \sigma & = 1.272015656 \\
 A & = -0.477617505 \\
 B & = 0.076216317 \\
 \Gamma & = 1.032427619 \\
 b_1 & = 0.477617505 \\
 b_2 & = 1.212078800 \\
 \gamma_1 & = -0.01512174 \\
 \gamma_2 & = 10.4 \mu m \\
 \eta_0 & = 1.6 \mu m \\
 \eta_1 & = 0.43 \mu m \\
 \eta_0 = -\gamma_0 4 / \cos \ i \\
\end{align*}
\]
The resulting deformation coefficients $b_1$, $b_2$, $Y$ are practically identical to those given in Section 3.7.1 of the ESTEC report, which were taken from the printouts of a computer run of the MTF program provided by AML. From this we may draw a few important conclusions:

1. The AML software package for MTF computations makes use of the same formulae for the deformation coefficients as given in the present note.

2. Since these formulae only correct the third-order aberrations and neglect possible effects of the secondary mirror obstruction, pupil asymmetry, and the inclination of the corrector, the instrument thus corrected is most likely not optimally corrected.

3. Nevertheless, as shown by the MTF calculations (which of course take into account all these effects), the performance of the nominal instrument, in terms of MTF and residual chromatic aberrations, is satisfactory for its purpose.

4. The formulae in this note can be used to design alternative solutions which are corrected to the same degree as is the present nominal instrument.

5.2. Modified solution with curved corrector and longer back focal distance

As a second example we derive a solution very similar to the nominal but with a paraxially curved corrector in order to reduce the maximum thickness of the figuring. We also require a slightly increased back focal length $\Delta = 0.133$ m for a more easily accessible focus (cf. MATRA study, Nov. 1979) and $68.5^\circ$ basic angle corresponding to $i = 17.125^\circ$. Otherwise we wish to retain the relative positions of the mirrors, i.e. $d = 0.700$ m and $x_1 = 1.300$ m, and the equivalent focal length should be approximately the same, $f_e \sim 2.5$ m. After a few trial solutions, the following input characteristics seem appropriate:

\[
\begin{align*}
  f_0 & = 667.87 \text{ m} \\
  f_1 & = 1.060 \\
  d & = 0.700 \\
  \Delta & = 0.133 \\
  x_1 & = 1.300 \\
  h_0 & = 0.160 \\
  i & = \pm 17.125^\circ \\
  \alpha & = 0.64^\circ.
\end{align*}
\]
We find:

(7) \( f'_1 = 1.060381049 \)
(8) \( x'_1 = 1.302064017 \)
\( d' = 0.702064017 \)
\( \Delta' = 0.130935983 \)
\( \sigma' = 0.337913462 \)
\( f''_e = 2.465128184 \)
\( f''_o = 0.628794601 \)
\( \eta'' = 0.418 \)
\( \gamma'' = 0.175 \)
\( \rho = 0.667 \)
\( \rho = 1.544909210 \)
\( \nu = 0.592989286 \)
\( \Omega = 0.393848178 \)
\( \nu = 2.179154748 \)
\( \sigma = -1.959337556 \)
\( \sigma = 1.227920867 \)

N.B. \( f''_e = (d' + \Delta')/\eta'' = (d + \Delta)/\eta' \)
N.B. \( f''_o = f''_e (f''_e - d'')/(f''_e - f''_o) \)

The residual aberrations of this modified system should be similar to those of the nominal instrument (Section 5.1), since no drastic changes have been introduced in the Gaussian parameters, \( f \)-ratio, or deformation coefficients. Inasmuch as the tilted corrector introduces uncorrected aberrations, the modified solution is even expected to be superior, thanks to the smaller departures of the correcting surface from a flat.
6. Flat-field Schmidt-Cassegrain systems

6.1. Formulae

We shall briefly investigate possible solutions with a flat focal surface, i.e. zero Petzval curvature. The curvature of the Schmidt corrector is neglected for simplicity. According to eq. (6) the flat-field condition is

\[ f_1 = f_2 \quad (= f). \]  

(21)

Eqs. (1) - (2) becomes

\[ f_e = f^2/d, \quad \Delta = f_e - f - d. \]  

(22), (23)

To find solutions compatible with the present mechanical constraints, we may regard d and \( \Delta \) as given, and solve for \( f \) and \( f_e \):

\[ f_e = \frac{3}{2} d + \Delta \pm (d(\frac{5}{4} d + \Delta))^\frac{1}{2} \]  

(24)

\[ f = (f_e d)^\frac{1}{2}. \]  

(25)

With the modified nominal values \( d = 0.700 \, \text{m}, \Delta = 0.133 \, \text{m}, \) and \( x_1 = 1.3 \, \text{m} \) we find \( f_e = 2.023 \, \text{m}, \eta = 0.493, \) and deformation thicknesses \( T_0, T_1, T_2 = 2.7 \, \mu\text{m}, 3.3 \, \mu\text{m}, 1.2 \, \mu\text{m}. \) More complete data are given in the table on next page. Thus it seems to be possible to make a flat-field instrument within the present mechanical envelope, with the following drawbacks however:

- the aspherical deformations especially on the primary and secondary mirrors are considerably larger than for the present baseline instrument or the modified instrument in section 5.2;
- the equivalent focal length is reduced from 2.5 m to about 2.0 m;
- the linear obstruction ratio is increased to about 0.50.

The reduced focal length is not a serious drawback, especially if the flat field means that a more accurate technique can be used for manufacturing the grid (e.g. Heidenhain rather than proposed Jobin-Yvon technique). The increased obstruction ratio has a negative effect on the photonstatistical astrometric accuracy (the increase of \( \eta \) from 0.45 to 0.50 corresponds to about 15% increased mean errors). The required asphericities may however be the most critical problem.
### Table: Main characteristics of some flat-field systems (α = 0.6°, i = ±17.12°, h₀ = 0.160 m).

<table>
<thead>
<tr>
<th>Solution</th>
<th>FF-1</th>
<th>FF-2</th>
<th>FF-3</th>
<th>FF-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>x₁</td>
<td>1.300</td>
<td>1.500</td>
<td>1.300</td>
<td>1.400</td>
</tr>
<tr>
<td>d</td>
<td>0.700</td>
<td>0.700</td>
<td>0.650</td>
<td>0.800</td>
</tr>
<tr>
<td>Δ</td>
<td>0.133</td>
<td>0.133</td>
<td>0.133</td>
<td>0.133</td>
</tr>
<tr>
<td>f₁=f₂</td>
<td>1.190000000</td>
<td>1.190000000</td>
<td>1.108948340</td>
<td>1.352050419</td>
</tr>
<tr>
<td>fₑ</td>
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<td>2.023000000</td>
<td>1.891948340</td>
<td>2.285050419</td>
</tr>
<tr>
<td>q</td>
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<td>0.413859080</td>
<td>0.408306089</td>
</tr>
<tr>
<td>n</td>
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<td>0.498007657</td>
<td>0.491617441</td>
<td>0.498082248</td>
</tr>
<tr>
<td>h₁</td>
<td>0.174</td>
<td>0.176</td>
<td>0.174</td>
<td>0.175</td>
</tr>
<tr>
<td>h₂</td>
<td>0.079</td>
<td>0.080</td>
<td>0.079</td>
<td>0.080</td>
</tr>
<tr>
<td>n*</td>
<td>0.577</td>
<td>0.546</td>
<td>0.558</td>
<td>0.595</td>
</tr>
<tr>
<td>b₁</td>
<td>1.543625233</td>
<td>1.139884660</td>
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<td>1.703011711</td>
</tr>
<tr>
<td>b₂</td>
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<td>11.06293732</td>
<td>12.39809560</td>
<td>15.28694355</td>
</tr>
<tr>
<td>γ</td>
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<td>-0.017715369</td>
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<tr>
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<td>2.22 μm</td>
<td>3.04 μm</td>
<td>2.01 μm</td>
</tr>
<tr>
<td>T₁</td>
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<td>2.52 μm</td>
<td>3.49 μm</td>
<td>2.50 μm</td>
</tr>
<tr>
<td>T₂</td>
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<td>1.03 μm</td>
<td>1.36 μm</td>
<td>0.97 μm</td>
</tr>
</tbody>
</table>

In connexion with the question of obstruction ratio we should note the following however. Presently, one of the more disturbing problems is the relatively large observational dead time due to inefficient baffling. This could be eliminated by re-introducing an internal baffle at the level of the secondary mirror. This increases however the effective obstruction ratio to some value \( n^* > n \). Eventually it may be found necessary to accept this increased obstruction. In that case the number \( n^* \) should be compared between different systems rather than \( n \). The obstruction ratio of the internal baffle can be computed by means of the formula (cf. Fig. 4)

\[
n^*h₀ = \frac{(h₁-h₂)(wx₁+h₀Δ)+(h₀-w)h₁d}{(h₁-h₂)(x₁+Δ)+(h₀-w)d}; \quad w = fₑ\tanα. \tag{26}
\]

We have \( n^* = 0.558 \) for the nominal instrument (Section 5.1) and \( n^* = 0.565 \) for the modified instrument (Section 5.2). \( n^* \) is given in the table above for some possible flat-field systems.
From a comparison of the different solutions in the previous table, we may conclude the following.

(1) The internal baffle obstruction ratio $\eta^*$ is generally not worse for the flat-field solutions than for the nominal and modified instruments with similar overall dimensions.

(2) The obstruction ratio $\eta^*$ may be slightly decreased either by increasing the total length $x_1$ (FF-2) or by moving the secondary closer to the primary (FF-3). The required mirror deformations decrease in the former case and increase in the latter case.

(3) The required deformations become smaller also if the primary mirror is moved away from the secondary and the corrector (FF-4). In this case the lengthening of the instrument may be less for the same reduction of deformations. On the other hand, the obstruction ratio is then significantly increased.

The feasibility of a flat-field Schmidt-Cassegrain system thus seems to depend on a number of questions:

- are the more severe mirror deformations acceptable?
- is it possible to increase the total length of the instrument?
- how does the astrometric accuracy depend on the effective obstruction ratio?
- is a flat field really an advantage?

Fig. 4. Rays defining the position ($x_b$) and height ($h_b = \eta^* h_o$) of internal baffle preventing light from complex mirror to enter the FOV directly.