Formulas for comparison of Copenhozen simulation results with theoretical predictions

1. Notation

$n_{star} = \text{number of stars on the sky}$

$n_{scan} = \text{number of scans}$

$n_{scan/star} = \text{number of scans (average) during which a particular star is observed}$

$n_{star/scan} = \text{number of stars in the strip of the sky covered by one scan}$

$n_{star/frame} = \text{average number of stars simultaneously in the combined FOV}$

$n_{obs} = \text{total number of grid-coordinate observations (case of "instantaneous direction")}$

$n_{angle/scan} = \text{number of angle measurements per scan (case of "pair steering")}$

$\beta = \text{FOV size, i.e. angle along one of the sides of the square FOV (rad)}$

$\tau = \text{mission duration in years}$

$k = \text{number of scans per day}$

$\delta = \text{rotating angle (from sun to z axis)}$

$k' = \text{no. of revolution around scan per year}$

$w_2 = \text{average speed of 2 axis in sky due to rotating law (rad/yr)}$

$\sigma_{\eta}, \sigma_{\delta} = \text{m.e. of grid coordinate measurements (masccic)}$

$\sigma_{\phi} = \text{m.e. of angle measurement (masccic)}$

$\sigma_{\phi} = \text{m.e. of star adiosion (per scan) (masccic)}$

$F_\beta, F_\lambda, F_{\rho\rho}, F_{\phi\phi}, F_\phi = \text{improvement factors (step 3) for } \beta, \lambda, \rho, \rho, \phi, \phi, \phi,$

$Q_{frame} = \text{fractional overlap of successive frames}$

$Q_{scan} = \text{fractional overlap of successive scans (at antinodes)}$
2. Basic relations between different numbers

Since the solid angle of the combined FOV is $2\Phi^2$ (approx.), we have

$$n_{\text{star/frame}} = \frac{\Phi^2}{2\pi} n_{\text{star}}.$$  \(1\)

The fraction of the sky covered by one scan is $\sin \frac{\Phi}{2} \approx \frac{1}{2} \Phi$; thus

$$n_{\text{star/scan}} = \frac{1}{2} \Phi n_{\text{star}}.$$  \(2\)

The number of scans is

$$n_{\text{scan}} = 365K \tau.$$  \(3\)

In order to have overlap by $Q_{\text{scan}}$ on successive scans, we must have

$$\frac{1}{365K \tau} \omega_2 = (1 - Q_{\text{scan}}) \Phi ;$$  \(4\)

hence

$$n_{\text{scan}} = \frac{\omega_2 \tau}{(1-Q_{\text{scan}}) \Phi}.$$  \(5\)

Since $\Phi^2$ is the probability that an arbitrary (given) star is in the ship of the sky covered by a scan, the average number of scans during which the star is seen is

$$n_{\text{ scans/star}} = \frac{\Phi}{2} n_{\text{scan}} = \frac{\omega_2 \tau}{2(1-Q_{\text{scan}})}.$$  \(6\)

We note that the speed $\omega_2$ of the $z$ axis in the sky is given by

$$\omega_2 = 2\pi \left[ 1 + (K - \frac{1}{2}) \sin^2 \frac{\Phi}{2} \right]^{1/2},$$  \(7\)

and is $\omega_2 \approx 28$ rad/hr almost independent of $\Phi$, if $K = 270^\circ / \Phi$. Thus, if $Q_{\text{scan}} = 0.5$, we have $n_{\text{scan/star}} \approx 28\tau$, independent of $\Phi$, $n_{\text{star}}$, $\Phi$, etc.
The total number of observations is (one ob. per star per frame):

\[ N_{\text{obs}} = N_{\text{frame/scan}} \cdot N_{\text{frames/scan}} \cdot N_{\text{scan}} \]  

With overlap \( Q_{\text{frame}} \) in successive frames, we have

\[ N_{\text{frame/scan}} = \frac{2\pi}{(1-Q_{\text{frame}}) \Phi} \]  

which, combined with (1) and (5), gives us

\[ N_{\text{obs}} = \frac{W_2 T}{(1-Q_{\text{scan}})(1-Q_{\text{frame}})} N_{\text{star}}. \]  

Eqs. (5) - (10) refer to the case of "instantaneous direction" observation. If we make angle measurements between pairs of stars, we assume the smallest number of angle measurements. This is obtained by connecting each star image to each of the \( N_{\text{star/scan}} \) stars (within an angle \( \Phi \) ahead on the great circle. Since there are \( 2N_{\text{star/scan}} \) images per scan, we have

\[ N_{\text{angles/scan}} = 2 N_{\text{star/scan}} \cdot N_{\text{star/frame}} \]

\[ = \frac{\Phi^2}{2\pi} N_{\text{star}} \]  

and the total number of angle measurements is

\[ N_{\text{angles}} = N_{\text{scan}} \cdot N_{\text{angles/scan}} = \frac{W_2 T}{2\pi (1-Q_{\text{scan}})} \frac{\Phi^2}{2\pi} N_{\text{star}} \]

3. Relation between obliquity \( \omega_0 \), and \( \sigma_1 \)

Let \( T \) be the total time available for observation per scan, and let \( \sigma_1(t) = \sigma_0 t^{-1/2} \) be the m.e. of a grid coordinate after integration over a time \( t \) (for stars of average magnitude). Consider first the measurement of instantaneous direction.
Since the number of observations per scan is
\[ \text{Obs/scan} = \frac{\text{Obs}}{\text{scan}} = \frac{\Phi}{(1 - 0.5\text{m/m}) \text{n/scan}} \]
we have
\[ \sigma_y = \sigma_0 \left( \frac{T}{\text{n/scan}} \right)^{-1/2} = \sigma_0 T^{-1/2} \sqrt{\frac{\Phi \text{n/m}}{1 - 0.5\text{m/m}}} \]

Hence, the time available for each observation is \( T / \text{Obs/scan} \), and consequently
\[ \sigma_y = \sigma_0 \left( \frac{T}{\text{n/scan}} \right)^{-1/2} = \sigma_0 T^{-1/2} \sqrt{\frac{\Phi \text{n/m}}{1 - 0.5\text{m/m}}} \]

Turning now to angle measurements, we have the available time per angle measurement \( T / \text{angle/scan} \); this time is divided between the two scans in the pair, who receive \( T / 2 \text{angle/scan} \) each. Thus, from grid coordinates, we determine the m.e. \( \sigma = \sigma_0 \left( \frac{T}{2 \text{angle/scan}} \right)^{-1/2} \), and the m.e. of the angle between them is
\[ \sigma = \sqrt{\sigma_x^2 + \sigma_y^2} = \sqrt{\sigma_0^2 + \sigma_0^2 T^{-1/2} \left( \text{angle/scan} \right)^{-1/2}} \]

According to ETEC report, section 3.8.3, the abscissa m.e. is then given by
\[ \sigma_x = \sigma_0 \sqrt{\frac{\Phi \text{n/scan}}{2 \text{angle/scan}}} \]
\[ = 2 \sigma_0 T^{-1/2} \sqrt{\frac{1}{2} \Phi \text{n/scan}} \]
\[ = \frac{\phi_0 T^{-1/2} \sqrt{2U}}{\sqrt{\Phi \text{n/scan}}} \]
\[ = \sigma_0 T^{-1/2} \sqrt{U \Phi \text{n/scan}} \]

where \( U = 1.7 \) according to 3.8.3.
Comparison of (16) and (14) gives the relation between $\sigma_x$ and $\sigma_y$:

$$
\sigma_x = \sqrt{(1 - Q_{frame}) U} \sigma_y.
$$

With $Q_{frame} = 0.5$ and $U = 1.70$ we have $\sigma_x = 0.922 \sigma_y$.

### Influence of $S$

The result (17) neglects the attitude uncertainty, reflected mainly by $S$. In fact the measurements of $S$ enter in two different ways in the strict solution: firstly, it determines the attitude around the $x$ and $y$ axes, and therefore the necessary projection factors for reducing grid coordinates $y$ to angular along the reference great circle. Secondly, if $S$ is comparable with $\sigma_y$, it contributes direct information in the automatic parameters, especially on the parallaxes, if $S$ is relatively small (since the parallax displacements are $\Delta y \propto S^2$ and $\Delta S \propto \sigma S$). However, in practice $S >> \sigma_y$, and the second effect is then probably negligible.

The attitude error which gives rise to the angle on grid coordinate measurement is

$$
\delta \varphi = \cos \gamma_2 \delta \alpha_2 \cos \beta_2 - \sin \gamma_2 \delta \beta_2
$$

(ESPEC report, section 3.8). This is a rotation around either optical axis. If a frame contains one star from each Fov, the rotation around the two optical axes are fixed (in each field) by the $S$-measurement in.
the other fields. Since the basic angle is less than 90°, we may expect the m.e. of the rotation component of each field to be approximately
\[ \sigma_f = \frac{\sigma_0}{\sin \gamma}. \]  
(18)

Eq. (3, 8, 10) of the Ester report then gives an expression for the angle measurement m.e. and hence the abrasion m.e.:
\[ \sigma_a^2 = \frac{U \text{Насадка} \cdot \sigma_a(0)}{2 \text{Насадка} \cdot \gamma} + (\Delta S_{\text{Нас}})^2 \frac{\sigma_p^2}{\sin \gamma}. \]
\[ \frac{U \text{Насадка} \cdot \sigma_a(0)}{2 \text{Насадка} \cdot \gamma} = \]
\[ = \left(1 - Q_{\text{frame}} \right) U \sigma_a^2 + \frac{\pi U}{12 \sin \gamma} \frac{1}{N_{\text{Нас}}} \frac{\sigma_p^2}{\Phi}. \]
\[ \sigma_a(0) \text{ is the angle m.e. in case } \sigma_f = 0, \text{ i.e. from (17), and we have used} \]
\[ (\Delta S_{\text{Нас}})^2 = \frac{1}{6} \Phi^2, \]  
(20)

assuming that the stars are uniformly distributed in \( \Phi \in \left[ \Phi_0, \Phi_1 \right] \), and also the relation
\[ \frac{N_{\text{Нас}}}{N_{\text{Нас}}} = \frac{N_{\text{Нас}}}{N_{\text{Нас}}} \frac{1}{\Phi}. \]
\[ \]  
(21)

Eq. (19), with \( Q_{\text{frame}} = 0.5, U = 1.70, \phi = 68.5 \), is the
only formula required for computing \( \sigma_a \) for the other simulations. It is quite remarkable that \( \Phi \) and

spherical mean parameter does not at all enter into the
formula.
5. Coefficients of improvement

Analysis of the various computations of $F_c$ by myself, AML and Vaghi shows that the coefficients (after averaging over the sky) depend very little on $K$ (as long as $K > 270/3$), initial conditions, uniform vs. non-uniform scanning, or synchrotron vs. amphoteric scanning. In fact, the only parameters which are of importance are $\delta$ and the average number of scans per star ($n_{\text{scans/\text{star}}}$) and (for proper motions only) the mission duration $T$. Thus we have in fact to sufficient accuracy the simple relations

$$F_\beta = 1.26 \left(\cos \delta\right)^{-\frac{1}{2}} \left(n_{\text{scans/\text{star}}}\right)^{-\frac{1}{2}}$$

$$F_\lambda = 1.55 \left(\sin \delta\right)^{-\frac{1}{2}} \left(n_{\text{scans/\text{star}}}\right)^{-\frac{1}{2}}$$

$$F_\mu = \frac{\sqrt{12}}{T} F_\beta$$

$$F_\nu = \frac{\sqrt{12}}{T} F_\lambda$$

$$F_{\omega} = 0.86 \left(\sin \delta\right)^{-1} \left(n_{\text{scans/\text{star}}}\right)^{-\frac{1}{2}}$$

(23)

The numerical factors 1.26, 1.55, and 0.86 are averages derived from the numerical experiments referred to above. The factor $\sqrt{12}/T$ for proper motion is theoretical (assuming uniform distribution of observations in time $T \in [0, T]$), but is verified within a few percent by the numerical experiments. It should be noted that the fractional dead time $t$ is not included in (23) or any of the formulas thus far; the correction factor for $t$, i.e. is of course $(1-t)^{-\frac{1}{2}}$.

The use of astrometric parameters are then $t \cdot F_c \cdot 1.26 \cdot (1-t)^{-\frac{1}{2}}$. 

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6. Comparison of theoretical results with computer simulations

Parameters: \( q_{\text{scan}} = 2 \), \( \theta_{\text{scan}} = 0.5 \), \( W_x = 2 \theta \), \( \chi_{\text{scan,ctm}} = 2 \Delta \theta \).

The table below gives the estimated m.e. \( \sigma_{5}, \sigma_{4}, \sigma_{30} \) and \( \sigma_{100} \) from (19) + (23); numbers in parentheses are corresponding results from simulation experiments. (*) = average over \( |\beta| = 20^\circ - 40^\circ \) only.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>( \theta )</th>
<th>( N )</th>
<th>( \eta )</th>
<th>( \tau )</th>
<th>( \delta )</th>
<th>( \sigma_{5,\text{exp}} )</th>
<th>( \sigma_{5,\text{th}} )</th>
<th>( \sigma_{4,\text{exp}} )</th>
<th>( \sigma_{4,\text{th}} )</th>
<th>( \sigma_{30,\text{exp}} )</th>
<th>( \sigma_{30,\text{th}} )</th>
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<td>200</td>
<td>26.0.5</td>
<td>30°</td>
<td>10</td>
<td>100</td>
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<td>6.16</td>
<td>4.1</td>
<td>6.1</td>
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<td>5.68</td>
<td>13.18</td>
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<tr>
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<td>40°</td>
<td>10</td>
<td>100</td>
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<td>4.05</td>
<td>5.44</td>
<td>4.7</td>
<td>5.5</td>
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<td>11.18</td>
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<tr>
<td>5a, 30° 2</td>
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<td>10</td>
<td>100</td>
<td>12.70</td>
<td>3.51</td>
<td>5.68</td>
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<td>(12.0)</td>
<td>(4.6)</td>
<td>(7.30)</td>
<td>(4.6)</td>
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<tr>
<td>6c, 15° 2</td>
<td>450 39.0.5</td>
<td>90°</td>
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<td>100</td>
<td>9.82</td>
<td>3.55</td>
<td>5.33</td>
<td>(4.0)*</td>
<td>(7.1)*</td>
<td>(4.0)*</td>
<td>(5.3)</td>
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<td>6c, 10° 2</td>
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<td>10</td>
<td>100</td>
<td>9.82</td>
<td>3.40</td>
<td>5.07</td>
<td>(4.0)*</td>
<td>(5.5)</td>
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<tr>
<td>10a-c, 30° 2</td>
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<td>30°</td>
<td>10</td>
<td>1000</td>
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Average pecho: | experimental m.e. | theoretical m.e. |
<table>
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<tr>
<td>(excluding 10a-c): 0.98</td>
<td>1.05</td>
<td>1.14</td>
</tr>
<tr>
<td>(dispersion) ±0.11 ±0.16 ±0.18</td>
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The results of the experiments with \( \sigma_{5} = 1^\circ \) (10a-c) show that eq. (19) probably overestimates the influence of \( \sigma_{5} \). Disregarding this experiment, there is a very good general agreement between the predicted m.e. and the m.e. found from the simulations. This shows that the basic principles used for extracting the m.e. of the astrometric parameters are probably correct.