HIPPARCOS will produce measurements of angles between stars along the scanning great circle having separations either less than the FOV diameter ($0.9^\circ$) or within this angle from the basic angle ($\gamma \pm 0.9^\circ$). After corrections for field distortion, tilt of the grid, etc., these angles are referred to the instantaneous great circle joining the two optical axes of the instrument (the principal great circle, PGC). Then, using the approximate attitude obtained from the star mapper and a priori star positions, the angles may be projected onto an arbitrary reference great circle (RGC), which is kept fixed during one or a few revolutions of the instrument (up to about $12^h$). The RGC is chosen to coincide approximately with the average PGC in that time interval, in order to minimize projection errors produced by inaccurate knowledge of the true PGC (attitude).

It is then possible to solve, by least squares or otherwise, the relative positions (abscissas) of stars along the RGC, together with the basic angle, scale value of the grid, and (possibly) other instrumental "constants". This solution constitutes the first step of the so-called three-step procedure outlined in the Phase A Report /5/.

The resulting average precision of star abscissas is obviously proportional to the average precision of angle measurements and to the square-root of the number of angle measurements per star, but depends also on the structure of observations and in particular the basic angle. It is known, for instance, that comparatively poor solutions are obtained if the basic angle is an integer fraction of $360^\circ$ (e.g., $60^\circ$, $72^\circ$, or $90^\circ$).

We present here an analytical study of the abscissa solution, with the three-fold purpose of (1) estimating the relation between the precisions of angle measurements and resulting abscissas, (2) obtain indications for an optimization of basic angle, and (3) provide a simple analytical step 1 solution which can possibly be used as a "quick-look" facility to monitor the performance of the instrument.
Definition of rigidity of step 1

If $\sigma_v$ is the average mean error of an angle measurement, $n$ the number of stars, and $m$ the number of angle measurements in a solution, we may assume that the mean error $\sigma_a$ of a star abscissa is

$$\sigma_a^2 = R \frac{n}{2m} \sigma_v^2,$$

(1)

where $R$ is a dimensionless number depending mainly on the structure of observations and the basic angle. The particular definition of $R$ in (1) is motivated by a simple consideration of the "ideal" observation structure: let $d a_i$ ($i = 0, 1, \ldots, n - 1$) be abscissa errors and $d v_j$ ($j = 1, 2, \ldots, m$) angle errors. The observation structure is defined by the functions $i(j)$ and $i'(j)$ telling which two stars are connected in the $j$:th angle measurement. The observation equations are then

$$d a_i(j) - d a_{i'}(j) = d v_j$$

(2)

(neglecting for the moment the errors of the basic angle and grid scale). The normal equation for star no. $i$ becomes

$$m_i d a_i = \sum_j \left[ \delta_{i(j)} d a_{i'}(j) + \delta_{i'(j)} d a_i(j) \right] = \sum_j \left[ \delta_{i(j)} - \delta_{i'(j)} \right] d v_j,$$

(3)

where $m_i = \sum_j \left[ \delta_{i(j)} + \delta_{i'(j)} \right]$ is the number of angle measurements concerned with star no. $i$. $\delta_{ij}$ is the Kronecker delta.

The "ideal" observation structure would result in completely decorrelated observation equations for which off-diagonal elements could be neglected. The solution for $d a_i$ is then simply the right member of (3) divided by $m_i$. Since the right member is the sum of $m_i$ quantities $d v_j$, each of variance $\sigma_v^2$, we find that the variance of $d a_i$ would be $\sigma_a^2 = \sigma_v^2/m_i$. But $\sum_i m_i = 2m$; hence $\langle m_i \rangle = 2m/n$ and $R = 1$ in eq. (1).
The number $R$ is therefore a measure of the non-rigidity of the step 1 solution, in the sense that $R > 1$ approaches unity the closer we get to the "ideal" observation structure.

**Analytical solution**

Suppose that the $n$ stars are roughly equispaced along the RGC and numbered $i = 0, 1, \ldots, n-1$ after increasing abscissa. Corresponding to the length of the field of view and the basic angle we may define two integers $f$ and $g$ such that star no. $i$ may be connected to $i + 1, i + 2, \ldots, i + f$ within the same field and to $i + g + 1, \ldots, i + g + f$ within the preceding field (all numbers taken modulo $n$). Thus, $2f/n$ should be slightly less than the field diameter, and $\pi(2g + f + 1)/n \approx \gamma$.

If $d\gamma$ is the error of the basic angle and $ds$ the grid scale error, we have for each long $(f - p)$ connection an observation equation of the form

$$da_{i+g+j} - da_i - jds - d\gamma = dv_{ij},$$

for $j = 1, 2, \ldots, f$. For the short connections we have

$$da_{i+j} - da_i - jds = dv_{ij}.$$

The normal equation for the unknown $da_i$ becomes

$$4f^2 da_i = \sum_{j=1}^{f} \left[ da_{i-g-j} + da_{i-j} + da_{i+j} + da_{i+g+j} \right] =$$

$$= \sum_{j=1}^{f} \left[ dv_{(i-j)j} - dv_{ij} + dv_{(i-g-j)j} - dv_{ij} \right],$$

while for $ds$ and $d\gamma$ we immediately get

$$ds = -\frac{2}{nf(f+1)} \sum_i \sum_j dv_{ij},$$

$$d\gamma = \frac{1}{nf} \sum_i \sum_j (dv_{ij} - dv_{ij}).$$
The system of normal equations (6) is a cyclic convolution of the unknowns, suggesting that Fourier methods can be used for its solution. A suitable linear constraint must however be introduced to eliminate the singularity due to the undefined abscissa zero point. The constraint \( \sum a_i d_a = 0 \) is automatically satisfied if \( d_a \) are represented by the finite Fourier series

\[
d_a = \sum_{l=1}^{n/2} \left[ a_1 \cos \left( \frac{2\pi l}{n} \right) + b_1 \sin \left( \frac{2\pi l}{n} \right) \right]. \tag{9}
\]

For brevity we shall use the notation \( c_i = \cos \left( \frac{2\pi i}{n} \right), \)
\( s_i = \sin \left( \frac{2\pi i}{n} \right). \) Similar expansions are obtained for \( d_{v_{ij}} \)
and \( d_{V_{ij}}:\)

\[
d_{v_{ij}} = \frac{1}{2} d_{0j} + \sum_{l=1}^{n/2} \left( d_{1j} c_i + e_{1j} s_i \right), \tag{10}
\]

\[
d_{V_{ij}} = \frac{1}{2} D_{0j} + \sum_{l=1}^{n/2} \left( D_{1j} c_i + E_{1j} s_i \right), \tag{11}
\]

where

\[
d_{1j} = \frac{2}{n} \sum_{i=0}^{n-1} d_{v_{ij}} c_i, \quad e_{1j} = \frac{2}{n} \sum_{i=0}^{n-1} d_{v_{ij}} s_i, \tag{12}
\]

and similarly for \( D_{1j}, E_{1j}. \) Insertion of (9) and (10) - (11) into the normal equations (6) and identifying the Fourier coefficients on each side yields a solution for the coefficients \( a_1, b_1:\)

\[
a_1 = \sum_j \left[ d_{1j} (c_j - 1) - e_{1j} s_j + D_{1j} (c_j + g) - E_{1j} s_j + g \right] \frac{1}{4f - 2 \sum_j \left[ c_j + c_j + g \right]}, \tag{13}
\]

\[
b_1 = \sum_j \left[ e_{1j} (c_j - 1) + d_{1j} s_j + E_{1j} (c_j + g - 1) + D_{1j} s_j + g \right] \frac{1}{4f - 2 \sum_j \left[ c_j + c_j + g \right]}. \tag{14}
\]

Successive application of (12), (13) - (14), and (9) produces the desired solution. The abscissas are in fact obtained as a cyclic convolution of \( d_{v_{ij}}, d_{V_{ij}}; \) the coefficients for this can be computed once and for all when \( n, f, \) and \( g \) are known.