Appendix 1

Spectral window for proposed observation scheme

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The influence of attitude jitter on the measurement of angles between stars can be appreciated when the observation scheme (i.e., timetable for the sampling and switching between the stars) and the jitter power spectrum are known.

To a grid approximation (viz., as long as the jitter amplitude is small compared to the grid period, $\Delta t$), the error of an angle measurement between two stars ($1$ and $2$), $\varepsilon_{12}$, is obtained as a linear combination of the attitude errors of the two stars, $\Delta \eta(1)$ and $\Delta \eta(2)$:

$$\varepsilon_{12} = \int u_1(t) \cdot \Delta \eta(1) \, dt + \int u_2(t) \cdot \Delta \eta(2) \, dt$$  \hspace{1cm}(1)

where $u_1$ and $u_2$ are weighting functions defined by the observation scheme. Since the attitude spot can only dwell at one star at a time, we have at every $t$ either $u_1(t) = 0$ or $u_2(t) = 0$. Also it can be assumed that $\Delta \eta(1) = \Delta \eta(2)$ because the main contribution to $\Delta \eta(1)$ comes from attitude jitter around the $z$ axis, which is the same for the two stars. It follows that (1) can be written:

$$\varepsilon_{12} = \int u(t) \cdot \Delta \eta(t) \, dt$$  \hspace{1cm}(2)

which is in fact the convolution of $u(t)$ and $\Delta \eta(t)$ evaluated with time lag $= 0$.

Consequently, the variance of $\varepsilon_{12}$ may be evaluated from:

$$\sigma_{\varepsilon_{12}}^2 = 2 \int_0^\infty W(f) \cdot P(f) \, df$$  \hspace{1cm}(3)

where $P(f)$ is the power spectrum of the stochastic process $\Delta \eta(t)$:

$$P(f) = \int R(\omega) \cdot \exp(-j \omega t) \, dt \quad (\omega = 2 \pi f)$$  \hspace{1cm}(4)

$$R(\omega) = \text{Exp} [\Delta \eta(t) \cdot \Delta \eta(t - \tau)] = \lim_{T \to \infty} \frac{1}{2T} \int_0^T \Delta \eta(t) \cdot \Delta \eta(t - \tau) \, dt$$  \hspace{1cm}(5)

and $W(f)$ is the spectral window associated with the filtering function $u(t)$:

$$W(f) = \left| \int_{-\infty}^{\infty} u(t) \cdot \exp(-j \omega t) \, dt \right|^2$$  \hspace{1cm}(6)

We shall moreover take advantage of the well-known fact that if $u(t)$ can be described as the convolution of a set of functions $u_k(t)$: $u(t) = u_1(t) * u_2(t) * ... * u_n(t)$, then $W(f)$ is the product of the corresponding window functions:

$$W(f) = W_1(f) \cdot W_2(f) \cdot ... \cdot W_n(f).$$  \hspace{1cm}(7)

We shall evaluate $u(t)$, and then $W(f)$, for a proposed observation scheme (Lindgren 71-03-21), described as follows:
(The sampling period $\Delta t = 1/2048$ sec is taken as time unit)

1. Sampling period = $\Delta t$

2. Period of signal modulation = $n_1 \Delta t$ (*8 in baseline scheme*)

3. Dwell time on first star ($A$) = $n_1 \Delta t$ ($2 \leq n_1 \leq 30$ in baseline)

4. Dwell time on second star ($B$) = $n_2 \Delta t$ ($2 \leq n_2 \leq 30$ in baseline)

where $n_1 \cdot (n_1 + n_2) \Delta t = n_2 \Delta t$ ($n_2 = 256$ in baseline)

5. Total time spent on pair $A/B = n_2 \Delta t$ (*2048 in baseline*)

It is assumed that $n_2$ is a multiple of $n_1$, which is a multiple of $n_1$, which is a multiple of 1.

The loss of one sample per dwell period is neglected.

**Sampling period**

It can be shown that, to first order, the effect of a finite sample period $\Delta t$ is simply to average $\Delta \eta(t)$ over this period. This is equivalent to convolving with the function

$$u_0(t) = \begin{cases} \frac{1}{\Delta t} & \text{for } -\frac{\Delta t}{2} < t < \frac{\Delta t}{2} \\ 0 & \text{otherwise} \end{cases}$$

**Modulation period**

The effects of jitter on this level actually depends on the method used to obtain the phase of the modulated signal from the counts.

We may consider two methods: Fourier analysis and maximum likelihood estimator. (The first one is easier to implement but the second is more accurate.)

**a. Fourier analysis**

Let the signal be, as function of time,

$$I(t) = I_0 \left\{ 1 + a_1 \cos \left[ 2\pi \left( vt - \Delta \eta(t) \right)/s - \varphi_1 \right] + a_2 \cos \left[ 4\pi \left( vt - \Delta \eta(t) \right)/s - 2\varphi_1 \right] \right\}$$

where $i = 1, 2$ for the two stars, $v = \text{spin rate}$, $s = \text{grid period}$, $\varphi_i = \text{phase of each star}$.

Then if the counts are represented by the step function $C_i(t)$ we have the Fourier estimator of $\varphi_i$:

$$\varphi_i = \varphi_0 + \arctan \frac{\int C_i(t) \sin \left[ 2\pi vt/s - \varphi_1 \right] dt}{\int C_i(t) \cos \left[ 2\pi vt/s - \varphi_1 \right] dt}$$

where $\varphi_0$ is an arbitrary reference phase (*e.g. a priori* phase).

Choosing $\varphi_0 = \varphi_i$ (*the true phase*), we obtain to a good approximation the phase error:

$$\hat{\varphi}_i - \varphi_i \approx \frac{2}{I_0 a_1 T} \left( \frac{\int C_i(t) \sin \left[ 2\pi vt/s - \varphi_1 \right] dt}{\int C_i(t) \cos \left[ 2\pi vt/s - \varphi_1 \right] dt} \right)$$
The expectation of $C_i(t)$ is (9) averaged over $[t - \frac{\Delta t}{2}, t + \frac{\Delta t}{2}]$ or, to first order, using notation $\delta q(t) \equiv (u_0 - \Delta q)(t)$,

$$E[C_i(t)] = \sum_{j=0}^{n-1} \{ 1 + a_1 \sin [3\pi t_j - \Phi_i] + a_2 \sin [4\pi t_j - 2\Phi_i] \} \frac{2\pi}{\delta q(t)} +$$

$$+ \sum_{j=0}^{n-1} \{ a_1 \sin [3\pi t_j - \Phi_i] + a_2 \sin [4\pi t_j - 2\Phi_i] \} \frac{2\pi}{\delta q(t)} \tag{11}$$

Inserting this in (11) we finally have the expected phase error:

$$E[\hat{\phi}_i - \Phi_i] = \frac{2\pi}{5} \cdot \frac{\Delta t}{4} \int \sin^2 [2\pi t \Phi_i - 2\Phi_i] \delta q(t) \, dt$$

$$\tag{13}$$

Because of the discretization in sampling intervals, (13) should really be written as a sum, with coefficients which are recurring periodically every $n_i$ sample. Moreover, the positional error corresponding to (13) is $\frac{\Delta t}{2\pi}$ times the phase error (13); hence the resulting weighting function for each modulation period is:

$$u_j(t) = \sum_{j=0}^{n_i-1} w_j \delta(t - j\delta t) / \sum_{j=0}^{n_i-1} w_j \delta(t - j\delta t) \tag{14}$$

When $w_j = \sin^2 [2\pi t_j - \Phi_i]$ for $t = j\delta t$.

Note that $\sum_{j=0}^{n_i-1} w_j = \frac{n_i}{2}$ \tag{15} for all $n_i \geq 3$, independent of $\Phi_i$.

2.2. Maximum likelihood estimator

The maximum likelihood estimate of $\Phi_i$ is the solution of

$$\int C_i(t) \frac{f(2\pi t \Phi_i - \Phi_i)}{J(2\pi t \Phi_i - \Phi_i)} \, dt = 0,$$  \tag{17}$$

where $f(\phi)$ is the model intensity function:

$$f(\phi) = 1 + a_1 \cos \phi + a_2 \cos 2\phi$$ \tag{18}$$

It can be shown that the bias of $\hat{\Phi}_i$ is to first order given by:

$$E[\hat{\Phi}_i - \Phi_i] = \frac{2\pi}{5} \int \frac{J(2\pi t \Phi_i - \Phi_i)}{J(2\pi t \Phi_i - \Phi_i)} \delta q(t) \, dt / \int \frac{J(2\pi t \Phi_i - \Phi_i)}{J(2\pi t \Phi_i - \Phi_i)} \delta q(t) \, dt \tag{19}$$

With discretization we find the weighting function for one modulation period:

$$u_j(t) = \sum_{j=0}^{n_i-1} w_j \delta(t - j\delta t) / \sum_{j=0}^{n_i-1} w_j \delta(t - j\delta t) \tag{20}$$

with

$$w_j = \frac{J'(2\pi t \Phi_i - \Phi_i)}{J(2\pi t \Phi_i - \Phi_i)}$$ \tag{21} for $t = j\delta t$.

In this case however, $\sum_{j=0}^{n_i-1} w_j$ is not completely independent of $\Phi_i$.\"
3-4. Dwell time on $A$ and $B$.

$U_{4}(t)$ expresses the effect of jitter on one modulation period ($n_1$ samples). The effect from $n_A$ consecutive such periods is obtained by averaging the errors of the $n_A$ periods. This is represented by convolution with the function

$$\sum_{k=1}^{n_A} \frac{1}{n_A} \delta(t-kn_1 \Delta t).$$

The same thing applies to the observation of the other star ($B$) during $n_B$ modulation periods. Taking finally the difference between the two resulting errors, which gives the error of the angle measurement from the $(n_A+n_B)n_1$ samples, we find the equivalent weighting function:

$$U_{2}(t) = \sum_{k=1}^{n_A} \frac{1}{n_A} \delta(t-kn_1 \Delta t) - \sum_{k=n_A+1}^{n_A+n_B} \frac{1}{n_B} \delta(t-kn_1 \Delta t).$$

(22)

5. Total time spent on pair of stars

Finally, averaging $n_3/n_2$ such switch periods gives us the weighting function:

$$U_{3}(t) = \sum_{k=1}^{n_3/n_2} \frac{n_3}{n_2} \delta(t-kn_3 \Delta t).$$

(23)

6. Computing $W(t)$

The total weighting function is

$$u(t) = u_x(t) + u_y(t) + u_z(t) + u_y(t).$$

(24)

The functions $u_{x} - u_{y}$ are schematically depicted in Figure 1.

Application of (6) on $u_{0} - u_{3}$ gives us the spectral windows corresponding to the different levels of the observation scheme:

$$W_{0}(f) = \left( \frac{\sin \left( \pi f \Delta t \right)}{\pi f \Delta t} \right)^2$$

(25)

$$W_{1}(f) = \left( \sum_{k=0}^{n_1} W_{0}' \sin (k \frac{\pi f \Delta t}{n_1}) \right)^2 + \left( \sum_{k=0}^{n_1} W_{0}' \sin (k \frac{\pi f \Delta t}{n_1}) \right)^2$$

(26)

$$W_{2}(f) = \left( \frac{\sin \left( \frac{n_1 f \Delta t}{n_A} \right)}{n_A \sin \left( \frac{n_1 f \Delta t}{n_A} \right)} \right)^2 + \left( \frac{\sin \left( \frac{n_1 f \Delta t}{n_B} \right)}{n_B \sin \left( \frac{n_1 f \Delta t}{n_B} \right)} \right)^2 - 2 \left( \frac{\sin \left( \frac{n_1 f \Delta t}{n_A} \right)}{n_A \sin \left( \frac{n_1 f \Delta t}{n_A} \right)} \right) \left( \frac{\sin \left( \frac{n_1 f \Delta t}{n_B} \right)}{n_B \sin \left( \frac{n_1 f \Delta t}{n_B} \right)} \right) \cos(\pi n_1 f \Delta t)$$

(27)

$$W_{3}(f) = \left( \frac{n_B}{n_3} \frac{\sin \left( \frac{\pi f n_3 \Delta t}{n_3} \right)}{\sin \left( \frac{\pi f n_3 \Delta t}{n_3} \right)} \right)^2$$

(28)

For the particular case where $n_A = n_B$, (23) reduces to

$$W_{2}(f) = \left( \frac{\sin \left( \frac{\pi f n_3 \Delta t}{n_3} \right)}{n_2 \sin \left( \frac{\pi f n_3 \Delta t}{n_3} \right)} \right)^2$$

(29)

The functions $W_{0}(f)$ - $W_{3}(f)$ are shown in Figure 2 for some values of $n_1$, $n_3$ and $n_A$. 
From the combination of $W_0 - W_3(f)$ it can be seen that the complete spectral window $W(f) = W_0(f) - W_1(f) - W_2(f) - W_3(f)$ is mainly composed of a sequence of band-passes, each of width
\[
\Delta f = \frac{1}{n_2 \Delta t} \approx 0.6 \text{ Hz} \quad \text{(with } \frac{1}{\Delta t} = 1200 \text{ Hz, } n_2 = 2048) \]
located at frequencies which are multiples of $\frac{1}{n_2 \Delta t} = 4.6875 \text{ Hz}$.

The following table lists the more important windows:

<table>
<thead>
<tr>
<th>$f \cdot n_2 \Delta t$</th>
<th>$f$ (Hz)</th>
<th>strength [height of $W(f)$]</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>4.6875</td>
<td>1.6</td>
</tr>
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<tr>
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</tr>
<tr>
<td>161</td>
<td>754.7</td>
<td>0.24</td>
</tr>
</tbody>
</table>

...etc.
Figure 1

\( u_0(t) \)
(Sampling period)

\( u_1(t) \)
(Modulation period)
\( n_i = 8 \), Fourier method
\( \phi_0 = 32^\circ \)

\( u_2(t) \)
(Switch period)

\( u_3(t) \)
(Complete cycle)
\( n_2 \Delta t = 2048 \Delta t = 0.218 \)
Figure 2. Spectral windows versus normalized frequency \( f \Delta t = f / f_{samp} \)

- \( W_0(t) \)
- \( W_1(t) \) with \( n_1 = 8 \)
  - Maximum likelihood \( (\varphi = 0) \)
  - Fourier transform \( (\varphi = 0) \)
- \( W_3(t) \) with \( n_2 = 2.56 \) and \( n_3 = 2.98 \)