Positional errors induced by photocathode inhomogeneities

L. Lindegren 1979 March 21

Summary

A simple analytical model for the astrometric effects of cathode sensitivity variations (inhomogeneities) is derived assuming that a certain smoothing of the variations can be realized by a slight defocussing of the star image on the photocathode (to a size of about 5 arcsec). Considering a subobservation of 0.1 sec duration, it is found that neither strong sensitivity gradients, blemishes of any size, or reasonable (although still pessimistic) random sensitivity variations can produce errors comparable with the random errors expected from the photon noise of a 9th magnitude star. It is concluded that photometric calibration of the photocathode is not necessary from the astrometric point of view.

1. Introduction

As pointed out at a recent meeting of the Space Astrometry Science Team (ESTEC, Feb 28-March 1), one of the remaining problem areas in connection
with the astrometry payload is the question of calibrating the instrument and in particular mapping the sensitivity of the dissector tube photocathode. Since we are mainly concerned about spatial frequencies corresponding to the fundamental grid period (about 1 arcsec), a full mapping of the FOV with required resolution would involve of the order of $4 \times 10^7$ pixels. Appropriate averaging (e.g. along slits and periodically normal to them) can certainly bring down the required amount of data by two orders of magnitude, but the calibration would still remain a serious problem.

Surprisingly, there seems to have been no numerical evaluation of the effects to be expected from cathode inhomogeneities, although the problem has been known for some time and is included in the analytical formulation of the IDT output provided by AML [Theoretical Study of the accuracy, Volume 1, Sections 3.6, 3.10, 3.12, 5.6-5.7, etc.]. The model described in the following section is the simplest possible one, and a number of approximations will be introduced for its application to different cases. Nevertheless, it should provide reasonable order-of-magnitude estimates of expected effects.

2. General assumptions; analytical model

Among the assumptions involved in the model, the following seem to be the most important:

A. The sensitivity (O.E.) of the photocathode is a unique function of position $(x,y)$ on the cathode, which we write in the form:
\[
\sigma(x, y) = \sigma_0 \left[ 1 + f(x, y) \right],
\]

in which \( \sigma_0 \) is the average A.E. and \( f(x, y) \) gives the relative variations. We may assume that this A.E. refers to some unspecified effective wavelength, and that the intensities introduced below are correspondingly integrated over the spectrum.

B. The star image is moving at constant and known speed \( v \) across the photocathode, the coordinates of the photocentre being

\[
\begin{align*}
x(t) &= x_0 + vt \\
y(t) &= y_0
\end{align*}
\]

at time \( t \). The initial position \( x_0 \) is the parameter to be estimated from the photoelectron counts.

C. Behind the grid, the spatially integrated intensity is sinusoidally modulated according to

\[
I(x) = I_0 \left[ 1 + a \cos(2\pi x/s) \right]
\]

where \( x \) is the position of the image photocentre on the cathode, and \( s \) is the grid period referred to the plane of the cathode.

D. A defocussed star image is formed at the photocathode. Its shape is similar to that of the entrance pupil, viz. rectangular with length \( l \) and width \( w \) (in \( x \) and \( y \)-directions, respectively). The central obstruction is ignored.
Moreover, the defocussed image has, independent of position \( x \), a homogeneous brightness \( I(x)/wl \).

The last two assumptions are by far the most critical ones, since the smoothing of \( f(x,y) \) produced by integrating over the defocussed image turns out to be the key to an acceptable level of perturbations. In fact, assumption E in particular is probably rather questionable, but it seems that a more rigorous treatment of that point would drastically complicate our problem.

Introducing the smoothed relative sensitivity

\[
F(x) \equiv \frac{1}{wl} \int_{-l/2}^{l/2} \int_{-w/2}^{w/2} f(x+\xi, y_0+\eta) \, d\eta \, d\xi
\]

we find that the expected photoelectron rate at time \( t \) is

\[
B(t) = \sigma_0 I_0 \left\{ 1 + a_1 \cos\left[ 2\pi(k_0 + ut)/s \right] \right\} \left\{ 1 + F(x_0 + ut) \right\}.
\]

Fourier analysis of the actual counts will produce an estimate \( \hat{x}_0 \) of the true parameter \( x_0 \). The bias of the estimate will be

\[
\beta = E[\hat{x}_0 - x_0] = -\frac{5}{2\pi} \arctan \frac{\int_0^T B(t) \sin(2\pi vt/s) \, dt}{\int_0^T B(t) \cos(2\pi vt/s) \, dt}.
\]

where \( t \in [0, T] \) is the observed time interval. Insertion of \( F(x) = \text{const} \) yields \( \beta = 0 \) as it should [we may assume that \( T \) is an integer multiple of \( s/v \)].
Putting \( \lambda_0 = 0 \) and assuming \( |b| \ll s \) and \( s \ll vT \) we obtain the approximation

\[
    b = -\frac{s}{\pi a, vT} \int_0^{vT} F(x) \left[ \sin(2\pi x/s) + \frac{1}{3} a, \sin(4\pi x/s) \right] \, dx \quad (7)
\]

which we shall in fact further simplify by neglecting the second term:

\[
    b = -\frac{s}{\pi a, vT} \int_0^{vT} F(x) \sin(2\pi x/s) \, dx . \quad (8)
\]

Eqs. (4) and (8) describe the effect of an arbitrary sensitivity variation \( f(x,y) \).

Since (4) is a simple convolution and (8) extracts a single Fourier component of \( F(x) \); it is in fact possible to express \( b \) very compactly in terms of the Fourier transform of \( f(x,y) \) (in fact this is partly done in Section 4); here, we shall however stick to the straightforward formulation of (4) and (8), and study the effects of different kinds of inhomogeneities: a linear gradient, a periodic variation, blemishes, and stochastic variations.

In the numerical examples, we shall invariably assume a grid period of \( s = 1'' \) (or 3 \( \mu \)m on the cathode), modulation \( a = 0.5 \), observation time \( T = 0.1 \) sec, which is about the typical time for uninterrupted tracking of one and the same star, and spin rate \( v = 150 \) arcsec/sec so that exactly \( vT/s = 15 \) periods of modulation are covered. The dimensions of the defocussed image are \( l = 5'' \) and \( w = 2.5'' \).
It should always be remembered that the mean error due to photon noise is about 20 mas/sec for a 9th magnitude star and observation time $T=0.1$ sec. (Lindgren 79-02-20).

3. The effect of a linear sensitivity gradient

A linear variation of sensitivity,

$$f(x, y) = f_0 + f_1 x,$$

remains the same after averaging over the defocussed image, $F(x) = f_0 + f_1 x$. Eq. (8) then gives us

$$\sigma = \frac{f_1 s_1}{2 \pi^2 a_1} \sin(2\pi x T/s).$$

Typical variations of sensitivity may be in the range ±10%. Assume as a worst case a variation from $f=0.9$ to $f=1.1$ over the distance $vT = 15'' = 45\mu m$. We find $f_1 = \frac{1}{25}$ arcsec$^{-1}$ and

$$\sigma = 1.4 \text{ marsec}.$$

4. The effect of a periodic variation of sensitivity

The most important harmonic of $f(x,y)$ is clearly the component involving $\sin(2\pi x/s)$. Assume then

$$f(x, y) = A \sin(2\pi x/s).$$
After smoothing according to (9) we have

\[ F(x) = A \frac{\sin(\pi e/s)}{\pi e/s} \sin(2\pi x/s). \] (13)

In principle, it would be possible to guarantee \( F(x) \equiv 0 \) by proper tuning of \( e \), but we shall ignore this and assume again the worst case \( \sin(\pi e/s) = 1 \). We find

\[ b = -\frac{A s^2}{2\pi^2 a, e}. \] (14)

Since \( |A| \leq 1 \), we obtain in fact an absolute bound for the effect:

\[ |b| \leq \frac{s^2}{(2\pi^2 a, e)} = 20 \text{ marsec}. \]

This corresponds however to an extremely artificial worst case; with all reasonable assumptions we should find the expected effects to be much smaller.

5. The effect of a blemish

Assume uniform sensitivity \( f(x, y) = 0 \) except in an area centered on \((\xi, \eta)\), where \( f(x, y) = -1 \) (i.e., no sensitivity). The length of the blemish is \( \lambda \), the width of the part of it overlapping with the strip on the cathode illuminated by the star image is \( \beta \) (see Fig. 1). Since we will have \( F(x) = 0 \) for \( x = 0 \) and \( x = vT \), we may re-write (8) in the form

\[ b = -\frac{s^2}{2\pi^2 a, vT} \int_0^{vT} F'(x) \cos(2\pi x/s) \, dx. \] (15)
We easily see that $F'(x) = 0$ except for two intervals of $x,$

\[
\left[ \frac{\pi}{2} - \frac{\lambda}{2}, \min \left( \frac{\pi}{2} + \frac{\lambda}{2}, \frac{\pi}{2} + \frac{\lambda}{2} \right) \right] \quad \text{and} \quad \left[ \max \left( \frac{\pi}{2} + \frac{\lambda}{2}, \frac{\pi}{2} - \frac{\lambda}{2} \right), \frac{\pi}{2} + \frac{\lambda}{2} \right]
\]

where $F'(x) = -\frac{\beta}{\ell w}$ resp. $\frac{\beta}{\ell w}.$

Performing the integration in (15) we find

\[
\ell = \frac{\beta s^2}{\ell w \pi^3 a, \nu T} \sin(2\pi \ell / s) \sin(\pi \ell / s) \sin(\pi \lambda / s).
\]

Since $\beta \leq w$ we have the bound

\[
\left| \ell \right| \leq \frac{s^3}{\pi^2 a, \nu T} = 0.9 \text{ arcsec}.
\]

Thus, no matter how big the blowish, it will not produce any significant perturbation.

6. The effect of a stochastic variation of sensitivity

If $F(x)$ is a stochastic function characterized by the moments

\[
E[F(x)] = 0, \quad E[F(x)F(x + \Delta x)] = \rho(\Delta x)
\]

We find by squaring eq. (8) and taking the expectation:

\[
E[\ell^2] = \frac{s^2}{\pi^2 a, \nu T^2} \int_0^\infty \int_0^\infty \rho(x' - x) \sin(2\pi x' / s) \sin(2\pi x / s) \, dx \, dx'.
\]

Changing to variables $\Delta x = x' - x$ and $\bar{x} = \frac{1}{2} (x' + x)$ and performing the integration with respect to $\bar{x},$ we obtain
\[
E[L^2] = \frac{2\alpha^2}{\pi^2 a_i^2 vT} \int_0^{\nu T} \left[ (1 - \frac{\Delta x}{\nu T}) \cos (2\pi \Delta x/s) + \frac{s}{\pi \nu T} \sin (4\pi \Delta x/s) \right] \rho(\Delta x) \, d(\Delta x)
\]

To proceed, we must make some assumption about the correlation function \(\rho(\Delta x)\). Since we are expecting that the high-frequency components of \(F(x)\) may be troublesome, we should assume that \(\rho(\Delta x)\) drops off as fast as possible for increasing \(\Delta x\). The shortest correlation length is obtained assuming that \(f(x,y)\) is essentially "white", leading to:

\[
\rho(\Delta x) = \begin{cases} \\
\frac{\alpha^2}{\nu l} \left(1 - \frac{|\Delta x|}{l}\right) & \text{for } |\Delta x| \leq l \\
0 & \text{for } |\Delta x| > l.
\end{cases}
\]

\(\alpha^2\) is a constant determining the power of the fluctuations \(f(x,y)\).

For the practically relevant case \(l \leq \nu T\) we obtain directly:

\[
E[L^2] = \frac{\alpha^2 s^4}{2\pi^4 a_i^2 \nu l^2 \nu T} \left[ 1 + \frac{5l}{4\nu T} - \left(1 - \frac{l}{\nu T}\right) \cos \frac{2\pi \Delta x}{s} - \frac{s}{\pi \nu T} \sin \frac{2\pi \Delta x}{\nu T} + \cdots \right],
\]

neglecting a higher-order harmonic term. The expression in brackets is at most slightly greater than 2; hence we have

\[
L_{\text{rms}} \approx \frac{\alpha s^2}{\pi^2 a_i \nu \sqrt{\nu T}}.
\]

To get a figure for \(\alpha\) we notice that \(\alpha^2/\Lambda\) is the variance of \(f(x,y)\) averaged over any area \(\Lambda\). Thus \(\alpha/\sqrt{\Lambda}\) is the pixel-to-pixel variation of relative sensitivity if the photocathode is divided
into pixels each of area $A$. A variation of $\Delta E$ by $\pm 5\%$ is mentioned as typical for the UV converters of IUE, having a pixel size of $37 \times 37 \mu m^2$. Since this apparently includes also large-scale variations (shading), it is probably safe to assume

$$\Delta x^2 \leq 0.05^2 \times 12^2 \text{ arcsec}^2 = 0.36 \text{ arcsec}^2$$ (24)

for the IDT of HIPPARCOS (where $37 \mu m$ corresponds to $12''$).

With this figure we obtain finally

$$\sigma_{rms} \leq 4 \text{ arcsec} \text{.}$$ (25)

Notice that this number improves with observing time $T$ in the same manner as the error due to photon noise. The noise in (25) corresponds to the photon noise of a star of magnitude 5.5.

**Conclusions**

Except for the highly improbable and artificial case of strong periodical variations of sensitivity, with a spatial frequency close to that of the grid, there is hardly any possibility of getting errors comparable to the photon-noise errors expected for a typical program star (magnitude 9). Therefore, provided that the smoothing effect of the defocussed image is a realistic assumption, there is hardly any need for calibrating the inhomogeneities or even avoiding blemishes.