On the possibility to measure parallactic displacements normal to the scan in Option A

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1. Introduction

At the first meeting of the Science Team (77-02-15), it was suggested by C. A. Murray that there is a loss of information in Option A because angles (and, in particular, parallactic displacements) are measured only along, and not perpendicular to the scan. This note aims at an examination of that suggestion. The approach we shall take is to assume that we have slits with inclinations \( k (k = \tan \theta \text{, where } \theta \text{ is the angle between slit and z axis}) \) and determine the optimum value of \( k \), i.e. the value which minimises the parallax variance. It is assumed that the only source of error is attitude jitter, the power spectrum of which is smooth and decreasing with frequency (although it is clear that in reality much power is concentrated at discrete frequencies).

2. The transit time at one slit

The attitude jitter is reflected in the irregular displacements \( \delta y(t) \) and \( \delta z(t) \) of the image away from the ideal image path. Let us assume that \( \delta y \) and \( \delta z \) are independent and described by power spectra \( P_y(v) \) and \( P_z(v) \), normalised in such a way that \( \sigma_y^2 = \int_0^\infty P_y(v) \, dv \), etc. At slit no. \( i \) the error in the transit time \( t_i \) will be \( \delta t_i = (-\delta y(t_i) + k \delta z(t_i))/v \), where \( v = 220"/s \) is the speed of the image in the field.

3. The position derived from transit times

The \( y \) and \( z \) coordinates of the image at some average time are

\[
y = y_o + \sum_i v t_i, \quad z = z_o + \frac{1}{k} \sum_i \delta z_i \frac{t_i}{v} ;
\]

the corresponding errors are

\[
\begin{align*}
Dy &= -\sum_i \delta y_i + k \sum_i \delta z_i, \\
Dz &= -\frac{1}{k} \sum_i \delta y_i + \sum_i \delta z_i.
\end{align*}
\]

The spectral window corresponding to the linear combination
\[ A(v) = \left( \frac{\sin(m\pi v/v_o)}{m \sin(\pi v/v_o)} \right)^2, \quad (m = \text{number of slits}), \]

whereas the linear combination \( \sum_i \delta y_i \), with alternately \( p \) slits inclined at \( +k \), \( p \) slits at \( -k \), etc., gives the spectral window

\[ B(v) = A(v) \tan^2(p\pi v/v_o). \]

Thus, the power spectrum of \( \Delta y \) and \( \Delta z \) are

\[ P_{\Delta y}(v) = A(v) P_y(v) + \frac{k^2}{2} B(v) P_z(v), \]

\[ P_{\Delta z}(v) = k^{-2} B(v) P_y(v) + A(v) P_z(v). \]

4. The distance between two images

Actually, we make only differential measurements, i.e. of angles \( \Delta y \) and \( \Delta z \) between two images which are (approximately) subject to the same instantaneous displacements due to attitude errors. If strictly simultaneous measurements of positions could be made, there would be no errors in \( \Delta y \) and \( \Delta z \). But because the grid pattern repeats with a finite period (2ps), it will be necessary to compare positions measured at times which may differ by up to \( \pm ps/v \). The power window corresponding to a difference with time lag \( \tau \) is

\[ 2(1 - \cos 2\pi \nu \tau). \]

Averaging over \( |\tau| \leq ps/v \) we obtain the window

\[ 2(1 - \sin(2\pi \nu/v_o)/(2\pi \nu/v_o)). \]

We then have

\[ P_{\Delta y}(v) = a(v) P_y(v) + \frac{k^2}{2} b(v) P_z(v), \]

\[ P_{\Delta z}(v) = k^{-2} b(v) P_y(v) + a(v) P_z(v), \]

where

\[ a(v) = 2(1 - \sin(2\pi \nu/v_o)/(2\pi \nu/v_o)) \left( \frac{\sin(m\pi v/v_o)}{m \sin(\pi v/v_o)} \right)^2, \]

\[ b(v) = a(v) \tan^2(p\pi v/v_o). \]
The functions $a(\nu)$ and $b(\nu)$ are shown in Fig. 1 and 2 for two different arrangements of $12$ slits ($p = 1$ and $p = 3$). It can be concluded that $p > 1$ is unfavourable in case the power spectra $P_y(\nu)$ and $P_z(\nu)$ are not white but dominated by low ($\leq \nu_o$) frequencies, since both $a(\nu)$ and $b(\nu)$ become transparent at low $\nu$.

If $p = 1$, $m \gg p$, and $P_y(\nu)$ and $P_z(\nu)$ decrease steeply above $\nu = \nu_o$ we can integrate $P_y(\nu)$ and $P_z(\nu)$ to get the variances

$$\sigma_{\Delta y}^2 = 2 T^{-1} (P_y(\nu_o) + k^2 P_z(\nu_o/2)),$$

$$\sigma_{\Delta z}^2 = 2 T^{-1} (k^{-2} P_y(\nu_o/2) + P_z(\nu_o)), $$

where $T = m/\nu_o$ is the total observing time for each star. (Note that two image dissectors are foreseen.)

To get further, let us assume $P_y(\nu) = P_z(\nu) = P_0 \nu^{-\alpha}$ for $\nu \geq \nu_o/2$. Then

$$\sigma_{\Delta y}^2 = 2 P_0 T^{-1} \nu_o^{-\alpha} (1 + 2^\alpha k^2),$$

$$\sigma_{\Delta z}^2 = 2 P_0 T^{-1} \nu_o^{-\alpha} (1 + 2^\alpha k^{-2}).$$

4. The weight of a parallax determination

The parallactic displacements parallel to and normal to the scan are $\tilde{\omega}.p ||$ and $\tilde{\omega}.p _\perp$, where (putting $R = 1$)

$$p || = \sin \beta \cos \beta_R \sin(\lambda_0 - \lambda_R) + \cos \beta \sin \beta_R \sin(\lambda - \lambda_0),$$

$$p _\perp = - \cos \beta R \cos(\lambda_0 - \lambda_R)$$

according to Murray (77-02-08).

If the parallactic displacements are measured with mean errors $\sigma_{\Delta y}$ and $\sigma_{\Delta z}$, respectively, the weight of a typical observation is

$$w = \frac{2}{\sigma_{\Delta y}^2} + \frac{2}{\sigma_{\Delta z}^2}.$$

With the approximations above this will be

$$w = T (2P_0)^{-1} \nu_o^2 \frac{2}{p _\perp^2} \left(\gamma(1 + 2^\alpha k^2)^{-1} + (1 + 2^\alpha k^{-2})^{-1}\right),$$
where \( Y = \frac{p_{\parallel}^2}{p_{\perp}^2} \).

Fig. 3 shows how \( \gamma(1 + 2^\alpha k^2)^{-1} + (1 + 2^\alpha k^{-2})^{-1} \) depends on \( k \) for \( \alpha = 2 \) and \( \gamma = 0.1, 0.25, 0.5, 1, \) and 2, and for \( \alpha = 1 \) and \( \gamma = 0.5 \) and 1. For \( \gamma = 1 \) we have (for \( \alpha > 0 \)) a minimum at \( k = 1 \) (± 45° slits), and maxima (= 1) at \( k = 0 \) and \( k = \infty \). Obviously, slits with \( k = \infty \) cannot be used, since they would give no transits at all (\( \nu_0 = 0 \)). Thus, \( k = 0 \) is optimal for \( \gamma = 1 \). Considering that \( k \gg 0 \) inevitably means that we have to increase \( s \) and hence decrease \( \nu_0 \), \( k = 0 \) will, in fact, be optimal for \( \gamma > 0.25 \) if \( \alpha \sim 2 \).

5. Estimation of \( p_{\parallel}^2 \) and \( p_{\perp}^2 \)

\( p_{\perp} \) is the same for each star in a particular orbit. To average \( p_{\parallel}^2 \) over one orbit we introduce the angle \( u \) in the orbit reckoned from the ascending node. We then have

\[ p_{\parallel} = \sin(\lambda_0 - \lambda_R) \sin u + \sin \beta_R \cos(\lambda_0 - \lambda_R) \cos u, \]

and obtain

\[ p_{\parallel}^2 = \frac{1}{2} \sin^2 \beta_R + \frac{1}{2} \cos^2 \beta_R \sin^2(\lambda_0 - \lambda_R), \]

\[ p_{\perp}^2 = \cos^2 \beta_R \cos^2(\lambda_0 - \lambda_R). \]

Both for inclined scanning (\( \beta_R = \text{constant} = \zeta \)) and for revolving scanning (\( \sin \beta_R = \sin \zeta \sin \phi, \sin(\lambda_0 - \lambda_R) = - \sin \zeta \cos \phi / \cos \beta_R \)), we obtain, averaging over the entire period of observation,

\[ p_{\parallel}^2 = \frac{1}{2} \sin^2 \zeta \quad \text{and} \quad p_{\perp}^2 = \cos^2 \zeta; \quad \text{hence} \quad \gamma = \frac{1}{2} \tan^2 \zeta. \]

We have \( \gamma > 0.25 \) for \( \zeta > 35^\circ \), for which \( k = 0 \) will be optimal if \( \alpha \sim 2 \).

6. Summary

It has previously been shown that Option A, with revolving scanning, gives satisfactory resolution of the five astrometric parameters. For a homogeneous sky coverage, and considering that too large an angle \( \zeta \) would give rise to other problems, a good choice of parameters appears to be \( \zeta = 40^\circ \) and \( \phi = 6.75\lambda_0 \). In that case it will not be advantageous to introduce inclined slits.