SPACE ASTROMETRY

A three-step procedure
for deriving positions, proper motions, and parallaxes of stars
observed by scanning great circles (Option A).

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Introduction. With five astrometric unknowns per star ($\Delta \lambda \cos \beta$, $\Delta \beta$, $\mu_\lambda \cos \beta$, $\mu_\beta$, and $\pi$) and about 50 000 stars (~1 per square degree), the total number of unknowns is at least 250 000. Although it is comparatively easy to set up the equation of condition for each measurement of an angle in the composite field of view, the large number of unknowns prohibits a brute-force approach to the solution of the problem. For instance, the computing time with a CDC-6600 computer for inversion of an n-by-n matrix is (D.C. Brown, p. 259 in Conference on Photographic Astrometric Technique, ed. Eichhorn, NASA CR-1825)

$$t = 1.2 \times 10^{-10} n^3 \text{ hr} \tag{1}$$

or about 200 years for $n = 2.5 \times 10^5$.

Below is described a procedure in three steps intended to give an equivalent solution with some reduction of the amount of computation. The three steps are:

1. For each complete great-circle scan, we solve for the angles between the stars in this scan, the basic angle, the field scale, and (if possible) the orientation of the grid.

2. By combination of all great-circle scans we can set up a system of equations for the zero-points of the great circles in a consistent system of coordinates on the celestial sphere.

3. Combining the solutions in step 1 and step 2 for all observations of a specific star, it is now a trivial matter to set up and solve the normal equations for the five astrometric unknowns for this star. This process is repeated for all the stars.

The order of the largest matrix to be inverted in each of these steps is:
in step 1: \( n \sim \) number of stars per scan \((\sim 360)\),

2: \( n \sim \) number of scans \((\sim 15,000 \text{ for } 2.5 \text{ years of observation})\),

3: \( n = 5 \).

With \( n = 15,000 \), Eq. (1) gives \( t = 17 \) days. However, it seems to be possible and even advantageous to treat several consecutive scans together in step 1, as the same star is normally observed during a number of consecutive scans (this is particularly the case with the inclined scanning, whereas with revolving scanning it often happens that a star is observed only during 2 - 3 consecutive scans). This gives a slight increase of the number of different stars per group of scans, but reduces very markedly the total computing time. For instance, if we have 5 scans per group, the computing time for the matrix inversion in step 2 reduces from 17 days to 3.3 hours.

**Step 1**

Assume that we know approximately the pole of the great circle \((\lambda_R', \beta_R')\) and the apparent positions \((\lambda_i, \beta_i)\) of the stars. Also, we have a nominal value \( \gamma_0 \) of the basic angle and of the scale \((s_0, \text{ in arc/distance})\) on the grid. From \((\lambda_R', \beta_R')\) and \((\lambda_i, \beta_i)\) we compute the angles \( v_i \) of the stars along the great circle, with an arbitrary zero-point. The actual basic angle is set as \( \gamma_0 + \delta \gamma \), the actual scale value as \( s_0 + \delta s \), and the actual angles of the stars as \( v_i + \delta v_i \), with the additional restriction

\[
\sum_i \delta v_i = 0 \quad \text{(summation over stars observed during this scan)},
\]

to secure a unique solution. (Uniqueness will also require that all the stars are directly or indirectly connected with all the other stars through angle measurements, and that at least one star is used to close the circle. I assume that these conditions are fulfilled.) Disregarding the misorientation of the grid and higher order terms (projection effects etc.), we obtain the equations of condition

\[
v_{i1} + \delta v_{i1} - v_{i2} - \delta v_{i2} = c(\gamma_0 + \delta \gamma) + (s_0 + \delta s)(y_{i1} - y_{i2}),
\]

where \( y_{i1} \) and \( y_{i2} \) are simultaneous coordinates of the two stars.
$i_1$ and $i_2$ in the grid system; $c = 0$ if the two stars are observed in the same field of view, $c = 1$ if $i_1$ is in the $p$-field and $i_2$ is in the $f$-field. With the assumed coordinates of the two stars, we would expect the distance in the grid system

$$(y_{i_1} - y_{i_2})_0 = (v_{i_1} - v_{i_2} - c\gamma_0)/s_0;$$

thus, with the notation $dy = (y_{i_1} - y_{i_2}) - (y_{i_1} - y_{i_2})_0,$

$$dv_{i_1} - dv_{i_2} - c. dy - (y_{i_1} - y_{i_2}) ds = s_0 dy. \quad (3)$$

If we number the angle measurements within this scan (or group of scans) $l = 1, 2, \ldots$, we can regard $i_1$ and $i_2$ as functions of $l$, and we can introduce $Y_l = y_{i_1}(1) - y_{i_2}(1)$. The complete set of equations of condition is then

$$dv_{i_1}(1) - dv_{i_2}(1) - c_1 dy - Y_l ds = s_0 dy, \quad l = 1, 2, \ldots,$$

$$\sum_i dv_i = 0 \text{ (summation over } i \text{ such that } i=i_1(1) \text{ or } i=i_2(1) \text{ for some } l). \quad (4)$$

The normal equations for the $m$ unknowns $dv_i$ and the $n$ additional unknowns $dy$, $ds$, etc. will have the following structure:

\[
d = m \cdot \frac{\text{Length of field}}{360^\circ}
\]

\[
g = m \cdot \frac{\gamma}{360^\circ}
\]
The unshaded areas of the \((m+n)^2\) matrix contains ones and need not be stored. The equations are perhaps most conveniently solved by Gauss-Seidel iteration, in which case it will never be necessary to store more than is contained in the shaded areas.

**Step 2**

We number the great-circle scans (or groups of scans) \(j = 1, 2, \ldots, J\) and we have to find \(J\) zero-points \(b_j\). Let us number the observations \(k = 1, 2, \ldots, K\). By the term 'observation no. \(k\)' I now mean a value \(\delta v_i\) obtained during step 1 for a star \((i)\) observed in scan no. \(j\); \(i\) and \(j\) may be regarded as functions of \(k\), and I denote the observation \(\delta v_k\), since the same star has been observed in completely different scans as well.

The equation of condition for observation no. \(k\) is

\[
\bar{P}_k \bar{R}_i(k) + b_j(k) = \delta v_k, \tag{5}
\]

where \(\bar{R}_i = (A\lambda_i \cos \beta_i, A\beta_i, \mu_{\lambda_i} \cos \beta_i, \mu_{\beta_i}, \pi_i)\) contains the astrometric unknowns for star no. \(i\), and

\[
\bar{P}_k = (-\sin \eta, -\cos \eta, -\tau \sin \eta, -\tau \cos \eta,\]

\[
+\bar{R}(\sin(\lambda - \lambda_\odot) \sin \eta + \sin \beta \cos(\lambda - \lambda_\odot) \cos \eta)) \tag{6}
\]

with the notation of Hòg, Appendix E.

Combining the \(K\) observations of the \(I\) stars during the \(J\) scans, we obtain formally a system of equations of condition which can be written

\[
P \times B = D
\]
or

\[ P R + Q B = D, \]  

(7)

where the elements of the matrices \( P, Q, R, B, \) and \( D \) are

\[ (P)_{ki} = \delta_{i,k} P_k, \]

\[ (Q)_{kj} = \delta_{j,k} (k), \]

\[ (R)_{i1} = \bar{R}_i, \]

\[ (B)_{j1} = b_j, \]

\[ (D)_{k1} = d_{v_k}. \]

The normal equations are formed in the usual manner, leading to

\[
\begin{pmatrix}
  P^t P & P^t Q \\
  Q^t P & Q^t Q \\
\end{pmatrix}
\begin{pmatrix}
  R \\
  B \\
\end{pmatrix}
= 
\begin{pmatrix}
  P^t D \\
  Q^t D \\
\end{pmatrix}
\]

or

\[ P^t P R + P^t Q B = P^t D, \]  

(9a)

\[ Q^t P R + Q^t Q B = Q^t D. \]  

(9b)

We note that both \( P^t P \) and \( Q^t Q \) are diagonal matrices (i.e., off-diagonal elements are all zero); the non-zero elements are, respectively,
\[(P^tP)_{ii} = \sum_{k(i)} \overline{P}^t_{k(i)} k^t k, \quad (10)\]

\[(Q^tQ)_{jj} = \sum_{k(j)} 1 = \text{(number of stars observed in scan no. } j) \quad (11)\]

Here, the symbols \(k(i)\) and \(k(j)\) under the summation signs mean that the summations are taken over all observations \(k\) concerned with star no. \(i\), or over all observations \(k\) made during scan no. \(j\), respectively.

The inverses \((P^tP)^{-1}\) and \((Q^tQ)^{-1}\) are also diagonal matrices with

\[\left[(P^tP)^{-1}\right]_{ii} = \left[\sum_{k(i)} \overline{P}^t_{k(i)} k^t k\right]^{-1} = \overline{S}_i \quad (5\times5 \text{ matrix}) \quad (12)\]

and

\[\left[(Q^tQ)^{-1}\right]_{jj} = 1/\left(\sum_{k(j)} 1\right). \quad (13)\]

Multiplication of (9a) to the left by \((P^tP)^{-1}\) gives

\[R + (P^tP)^{-1}P^tQ B = (P^tP)^{-1}P^tD, \quad (14)\]

and elimination of \(R\) between (14) and (9b) results in

\[(Q^tQ - Q^tP(P^tP)^{-1}P^tQ) B = Q^tD - Q^tP(P^tP)^{-1}P^tD. \quad (15)\]

The \(J^2\) matrix \(T = (Q^tQ - Q^tP(P^tP)^{-1}P^tQ)\) is singular, since we have not specified the coordinate system to which the unknowns should be referred. There are six degrees of freedom, corresponding to the six constants \(\theta_0\) and \(\dot{\theta}\) in the expressions \(\theta = \theta_0 + \dot{\theta}t\) for the rotations of the system around three axes. There are several ways to fix a preliminary coordinate system and make possible the solution of \(B\) from (15). One is simply to postulate \(b_j = 0\) for six different scans. Another is to use the pseudo-inverse of \(T\), whereby a solution is obtained for which \(\sum_j b_j^2\) = minimum. The pseudoinverse approach is attractive because it also provides good estimates of the variances of the \(b_j\)'s, but it requires more computation.
With the definitions (9) and (12) we find that

\[ \sum_{k(j_1)} F_k S_i(k) F_k'(i(k), j_2), \]

where \( k'(i(k), j_2) \) means an observation of star no. \( i(k) \) made during scan no. \( j_2 \); if no such observation exists, the contribution to the sum from the observation \( k \) is zero. It is seen that the element \( j_1 j_2 \) is zero if the two scans \( j_1 \) and \( j_2 \) have no star in common. For \( j_1 = j_2 \) we have \( k'(i(k), j_2) = k(j_1) \) and obtain

\[ \sum_{k(j)} F_k S_i(k) F_k^t. \]

The elements of the right-hand bracket in (15) are

\[ \sum_{k(j)} \left( dv_k - F_k S_i(k) \sum_{k'(i(k))} F_k' dv_{k'} \right). \]

**Step 3**

Returning to Eq. (5) we now know the \( b_j \)s and have for star no. \( i \):

\[ F_k(i,j) R_i = dv_k(i, j) - b_j, \]

for a number of different \( j \)s. The least-squares solution of this is

\[ R_i = S_i \sum_{k(i)} F_k (dv_k - b_j(k)), \]

which could also have been derived directly from (14):

\[ R = (F^t F)^{-1} F^t (P - Q). \]

**Summary**

By eliminating \( R \) in the normal equations (9) we have reduced the \((5I + J)^2\) system (9) to the \( J^2 \) system (15). This is still very large but it is not completely unrealistic to solve it, particularly if grouping of scans can be used.