Principles of Astrometry

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**astrometry, n.**

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Pronunciation:  Brit. /ˈastrəmətri/,  U.S. /ˈastrəmətri/

Frequency (in current use):  


**Astron.**

The measurement and analysis of the position, motion, or magnitude of celestial objects, esp. stars.

Quot. 1847  represents the introduction of the word in its modern technical use.

1811  *Philos. Mag.* **37** 45  No one has ever said that astronomy is astrometry or uranometry, although the heavens and the stars are measured.

1847  J. F. W. HERSCHEL  *Results Astron. Observ.* iii. 304  Of astrometry, or the numerical expression of the apparent magnitudes of the stars.
Astrometry → Directional measurements

(inner) solar system: a well-mapped space - distances, motions, and masses are very accurately known

Barycentric Celestial Reference System (BCRS): X, Y, Z, T [m, s]
T = barycentric coordinate time (TCB)

measured direction $u(T)$ (corrected for local effects)

source

photon path

International Celestial Reference System (ICRS)

here be dragons

Reference frames → F. Mignard
Relativistic models → S. Klioner
Source model (stellar and extragalactic objects)

“Source” = any sufficiently point-like object

**Model:** Constant space velocity in the barycentric system:

\[ b(T) = b_0 + (T - T_{ep})v \]

\( T_{ep} = \) reference epoch (e.g. J2015.0 for TGAS)

Model has 6 kinematic parameters:

\((b_{0X}, b_{0Y}, b_{0Z}, v_X, v_Y, v_Z) \iff (\alpha, \delta, \omega, \mu_{\alpha*}, \mu_{\delta}, v_R)\)

For the modelling, \(v_R\) can be ignored except for some very nearby stars

\( \Rightarrow \) 5 astrometric parameters:

standard model for “single” stars, quasars, etc
Why is the 5-parameter model good enough?

- Galactic orbits are curved $\implies$ negligible
- Variable surface structures $\implies$ significant only for some (super)giants
- Most stars are members of double/multiple systems $\implies$ curved motion

Period distribution of G dwarf primaries (Duquennoy & Mayor, 1991):

50% have a stellar companion
log-normal P with median = 180 yr and sigma = 2.3 dex

- $P < 10$ d: orbit $\ll$ parallax
- $P > 100$ yr: curvature $\ll$ parallax

40% of binaries have $10$ d $< P < 100$ yr $\implies$ 20% of sources will be problematic
Instrument (calibration) models

or

The limits of self-calibration
Calibration models

No universal model – depends entirely on the application:

- type of instrument
- wavelength region
- imaging or interferometric
- relative or absolute
- small-field or global
- space or ground-based
- ...

...
Example: Classical plate model

Used e.g. in photographic wide-field astrometry (AC, AGK2, AGK3, ...)

1. Measure plate coordinates \((x, y)\) of all objects
2. Identify “reference stars” with known \((\alpha, \delta)\)
3. Fit plate model \((x, y) \leftrightarrow (\alpha, \delta)\) to the ref. stars
4. Apply \(f\) to measured \((x, y)\) of the other objects

Problems:
- Low density of reference stars
- Higher-order models not possible
- Calibration not better than the reference stars
Plate-overlap technique (block adjustment)

Eichhorn (1960)

- Fit several overlapping plates simultaneously
- Every star measured on two or more plates gives additional constraints (for consistent $\alpha$, $\delta$)
- Need to solve large systems of equations
Self-calibration principle

1. Rely as little as possible on external “standards” – they are often not as good as your data!

2. Take multiple exposures of the same field at different times, orientation, etc.

3. Use parametrized models of sources \((s)\) and other relevant factors, e.g. telescope pointing and distortion (“nuisance parameters”, \(n\))

4. Solve the parameter values that best match the model \((f)\) to the data:

\[
\min_{s, n} \| \text{obs} - f(s, n) \|_M \Rightarrow s, n
\]

5. Usually, the solution is not unique \((s \in S_f = \text{solution space})\), and external standards may be used to select the preferred solution in \(S_f\)
Self-calibration example: HST cameras

- Anderson & King (2003) PASP 115, 113 (calibrating WFPC2 using $\omega$ Cen)

Pattern of exposures

Map of 89,000 stars used
Self-calibration example: HST Fine Guidance Sensors (FGS)

Calibration parameters:
- $\rho_A$, $\rho_B$ (arm lengths)
- $\kappa_A$, $\kappa_B$ (offsets in $\theta_A$, $\theta_B$)
- $a_{ij}$, $b_{ij}$ (distortion)

$x' = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{02}y^2 + a_{11}xy + a_{30}x(x^2 + y^2) + a_{21}(x^2 - y^2) + a_{12}(y^2 - x^2) + a_{03}y(y^2 + x^2) + a_{50}x(x^2 + y^2)^2 + a_{41}y(y^2 + x^2)^2 + a_{32}x(x^4 - y^4) + a_{23}y(y^4 - x^4) + a_{14}x(x^2 - y^2) + a_{05}y(y^2 - x^2) + a_{21}(x^2 - y^2)$

$y' = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{02}y^2 + b_{11}xy + b_{30}x(x^2 + y^2) + b_{21}(x^2 - y^2) + b_{12}(y^2 - x^2) + b_{03}y(y^2 + x^2) + b_{50}x(x^2 + y^2)^2 + b_{41}y(y^2 + x^2)^2 + b_{32}x(x^4 - y^4) + b_{23}y(y^4 - x^4) + b_{14}x(x^2 - y^2) + b_{05}y(y^2 - x^2) + b_{21}(x^2 - y^2)$

Calibration field in M35
(McArthur, Benedict & Jefferys, 2002)
A simple toy model for illustration

- Superficially resembling the HST camera calibration
Neglecting $\nu_R$ the 5-parameter model is linear in tangential coordinates $\xi, \eta$ (gnomonic projection):

$$\xi(t) = a + bt + \omega \Pi_\xi$$
$$\eta(t) = d + et + \omega \Pi_\eta$$

$\Pi_\xi, \Pi_\eta$ = known parallax factors (assumed constant over the field)

$\rightarrow$ 5 parameters per source: $a, b, d, e, \omega$
Toy model: Calibration

Assume the most general linear relation between
• tangent plane coordinates \((\xi, \eta)\) and
• pixel coordinates \((x, y)\):

\[
\begin{align*}
x &= A + B\xi + C\eta \\
y &= D + E\xi + F\eta
\end{align*}
\]

\(\rightarrow\) 6 parameters per exposure: 
\[A, B, C, D, E, F\]
Toy model: Synthesis

$M$ stars ($i = 1...M$) in $N$ exposures ($j = 1...N$) $\rightarrow$ $2MN$ non-linear equations:

$$x_{ij} = A_j + B_j(a_i + b_it_j + \omega_i\Pi_{\xi_j}) + C_j(d_i + e_it_j + \omega_i\Pi_{\eta_j})$$
$$y_{ij} = D_j + E_j(a_i + b_it_j + \omega_i\Pi_{\xi_j}) + F_j(d_i + e_it_j + \omega_i\Pi_{\eta_j})$$

Linearisation gives a system of $2MN$ equations for $5M + 6N$ parameters ($\theta$):

$$J \times \Delta \theta = \text{obs} - \text{calc}, \quad \text{with Jacobian} \quad J = [\partial(\text{calc})/\partial\theta]$$

$$\text{rank}(J) < 5M + 6N \quad \rightarrow \quad \text{solution is not unique}$$

What is the rank, and what does it mean?
Toy model: Numerical simulation

Numerical simulation with
\[ M = 200 \text{ stars} \]
\[ N = 20 \text{ exposures} \]
randomly distributed over 2 years

\[ \rightarrow 8000 \text{ equations} \]
\[ 1120 \text{ parameters} \]

Compute \( J \) and make SVD
(Singular Value Decomposition)
Toy model: Singular values of $J$ (with 1120 parameters)

<table>
<thead>
<tr>
<th>Parameter index</th>
<th>Singular value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$10^{-15}$</td>
</tr>
<tr>
<td>1</td>
<td>$10^{-14}$</td>
</tr>
<tr>
<td>2</td>
<td>$10^{-13}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>1110</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>1115</td>
<td>$10^{-6}$</td>
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<tr>
<td>1120</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>1125</td>
<td>$10^{-8}$</td>
</tr>
</tbody>
</table>

Rank = 1105, nullity = 15
Toy model: Interpretation

Nullity = 15 \Rightarrow \text{ the solution has 15 degrees of freedom (degeneracies)}

Assume \[ \begin{bmatrix} s \\ n \end{bmatrix} \] is a least-squares fit of the models to the data \((s \in S_f)\).

Then \[ \begin{bmatrix} s + \Delta s \\ n + \Delta n \end{bmatrix} \] is an equally good fit, provided that \[ \begin{bmatrix} \Delta s \\ \Delta n \end{bmatrix} \] can be written as a linear combination of the 15 singular vectors with singular values \(\approx 0\).

Why 15?
The 15 singular vectors for the toy model (# 1)

position               proper motion              parallax

Only $\Delta s$ shown, but in each case there is an exactly "compensating" $\Delta n$
The 15 singular vectors for the toy model (#2)

position

proper motion

parallax

Only $\Delta s$ shown, but in each case there is an exactly “compensating” $\Delta n$
The 15 singular vectors for the toy model (# 3)

- position
- proper motion
- parallax

Only $\Delta s$ shown, but in each case there is an exactly “compensating” $\Delta n$. 
The 15 singular vectors for the toy model (# 4)

position  proper motion  parallax

Only $\Delta s$ shown, but in each case there is an exactly "compensating" $\Delta n$
The 15 singular vectors for the toy model (# 5)

position  proper motion  parallax

Only $\Delta s$ shown, but in each case there is an exactly “compensating” $\Delta n$
The 15 singular vectors for the toy model (# 6)

position  proper motion  parallax

Only $\Delta s$ shown, but in each case there is an exactly “compensating” $\Delta n$
The 15 singular vectors for the toy model (# 7)

position  proper motion  parallax

Only $\Delta s$ shown, but in each case there is an exactly “compensating” $\Delta n$
The 15 singular vectors for the toy model (# 8)

Only $\Delta s$ shown, but in each case there is an exactly "compensating" $\Delta n$.
The 15 singular vectors for the toy model (# 9)

position  proper motion  parallax

Only $\Delta s$ shown, but in each case there is an exactly “compensating” $\Delta n$
The 15 singular vectors for the toy model (# 10)

position  proper motion  parallax

Only $\Delta s$ shown, but in each case there is an exactly “compensating” $\Delta n$
The 15 singular vectors for the toy model (# 11)

position  proper motion  parallax

Only $\Delta s$ shown, but in each case there is an exactly “compensating” $\Delta n$
The 15 singular vectors for the toy model (# 12)

position  proper motion  parallax

Only $\Delta s$ shown, but in each case there is an exactly “compensating” $\Delta n$
The 15 singular vectors for the toy model (# 13)

position  proper motion  parallax

Only $\Delta s$ shown, but in each case there is an exactly “compensating” $\Delta n$
The 15 singular vectors for the toy model (# 14)

position    proper motion    parallax

Only $\Delta s$ shown, but in each case there is an exactly “compensating” $\Delta n$
The 15 singular vectors for the toy model (# 15)

position

proper motion

parallax

Only $\Delta s$ shown, but in each case there is an exactly “compensating” $\Delta n$
Implications of the model degeneracies

Every $\Delta s$ in the solution space has a compensating $\Delta n$ (and vice versa)

Hence degeneracies -

• *could hide actual astrophysical patterns in* $s$
  - the patterns are absorbed by $n$ instead

• *could hide actual instrumental effects in* $n$
  - instead, the effects become systematic errors in $s$

• *could be difficult to discover in complex problems*
  - in particular, none of the problems above would show up in the residuals
Dealing with the degeneracies

A few possible strategies:

1. **Accept as a practical limitation (“relative astrometry”)**
   → Important to know and understand the solution space

2. **Constrain the source parameters**
   → E.g. use quasars for the zero point of proper motion and parallax

3. **Constrain the nuisance parameters**
   → E.g. use laser metrology to fix some calibration parameters

4. **Use a different technique**
   → E.g. global astrometry can eliminate many degeneracies in relative astrometry
Self-calibration for Hipparcos and Gaia

The **Gaia astrometric global iterative solution** uses a block-iterative method to solve

\[
\min_{s, a, c} \| \text{obs} - f(s, a, c) \|_M
\]

- nuisance parameters are the attitude \((a)\) and geometric calibration \((c)\)

A similar method was used for the Hipparcos re-reduction (van Leeuwen 2007)

<table>
<thead>
<tr>
<th>Number of parameters (millions)</th>
<th>(s)</th>
<th>(a)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Hipparcos</strong></td>
<td>0.5</td>
<td>1</td>
<td>0.05</td>
</tr>
<tr>
<td><strong>Gaia DR1</strong></td>
<td>10</td>
<td>1.5</td>
<td>0.1</td>
</tr>
<tr>
<td><strong>Gaia (final)</strong></td>
<td>100</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

(COUNTING ONLY THE PRIMARY SOLUTION AND ALONG-SCAN DATA)
The limits of self-calibration

- The astrometric solutions for Hip and Gaia involve millions of parameters.

- Some degrees of freedom are well known and explicitly taken care of in the solutions (e.g., the reference frame).

- Can we confidently say we know and understand all the degrees of freedom?

- Numerical simulations are helpful: SVD may not be feasible, but one can generate random vectors \((\Delta s, \Delta n)\) in the solution space.
Conclusions

Self-calibration is great but cannot determine everything!

→ For interpreting the results one needs to know the solution space $S_f$
→ This depends on the models used ($f$), not on the data

Very careful attention should be given to the calibration models in complex projects such as Gaia

→ Unrecognised degrees of freedom could produce systematics that are not revealed by the residuals
→ Numerical simulations may be the only practical way to explore possible weaknesses in the solution